David Gale's Subset Takeaway Game

You can go first, or you can go second; but will you win?

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Abstract

In this expository paper, we investigate David Gale's *Subset Takeaway Game*, and discuss results, conjectures, and generalizations related to this game. In particular, we describe the natural interpretation of the game in terms of simplicial complexes, and then we desribe the binary star reduction technique that shows that Subset Takeaway is a second player win for $n \leq 6$. We also look at Subset Takeway played on a graph, and compute the Grundy values for complete n-partite graphs and all bipartite graphs.

1 Introduction

David Gale's Subset Takeaway is a combinatorial game similar to CHOMP that has puzzled mathematicians for decades. It has an innocent-looking formulation, but it turns out to be devilishly hard to study. Let's start by looking at its definition.

Definition 1 (Subset Takeaway)**.** Let $n \in \mathbb{N}$, and let $X = \{1, \ldots, n\}$. Subset takeaway is an impartial game played on the power set 2^X consisting of all subsets of X. In a move, a player deletes some of the subsets, and as usual, the player who cannot make a move loses. The allowable moves are as follows:

> *A player picks a proper, non-empty subset $S \subset X$ that has not been deleted. Then S is deleted, and all subsets of X (that haven't already been deleted) that contain S are deleted.*

In the definition, S is chosen to be non-empty because otherwise the first player may just take the empty set and win. Using a strategy-stealing argument, we can also show that if S is allowed to equal X, then the first player will always win. From now on, we refer to the first player as Alice, and the second player as Bob.

Example 1. We can manually work out the outcome of the game for the cases $n = 1, 2, 3$.

For $n = 1$, Bob wins, because $2^X = \{\varnothing, X\}$, and there are no proper subsets that Alice could move to.

For $n = 2$, Bob wins, because $2^X = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$, and Alice can pick one of $\{1\}$ or $\{2\}$, and then Bob can pick the other to win.

Finally, let's look at $n = 3$. Bob wins here as well. We have

$$2^X = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Up to symmetry, Alice has two possible moves: $\{1\}$ or $\{1,2\}$. The possible outcomes are

$$\{\varnothing, \{2\}, \{3\}, \{2,3\}\} \quad \text{and} \quad \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$$

In the first case, suppose Bob picks $\{2,3\}$. Then the position becomes $\{\varnothing, \{2\}, \{3\}\}$. This is clearly a second player win, because if Alice picks $\{2\}$, Bob can pick $\{3\}$ and vice versa. The second case is also a second player win: suppose he picks $\{3\}$ and the position becomes $\{\varnothing, \{1\}, \{2\}\}$. Then just as before, this is a winning position for him.

So we have found the following:

Observation 2. *Subset Takeaway is a second player win for* $n = 1, 2, 3$. *Additionally, Bob's winning move is to take the complement of Alice's set.*

This turns out to be true for $n = 4, 5, 6$ as well, for instance see Christensen and Tilford [2].

Theorem 3. *Subset Takeaway is a second player win for* $n \leq 6$. *Additionally, Bob's winning move is to take the complement of Alice's set.*

From this, Gale made the conjecture that Subset Takeaway is always a second-player win. Additionally, Christensen and Tilford conjectured more recently that a winning response to an opening move is to play the complementary move.

However, it turns out to be very hard to study this game in the form which we have presented it. To make any more progress, we need a new idea. Surprisingly, this idea turns out to be to use the tools of topology.

2 Simplical Complexes

A simplex is a generalization of triangles and tetrahedrons into higher dimensions. Specifically, a $(n-1)$-simplex is a collection of n vertices $1, \ldots, n$, which have faces between them. A face consists of some (non-empty) subset of these vertices; and there is a a face corresponding to each subset. For example, a 2-simplex is a triangle since it has 3 vertices $1, 2, 3$. The "faces" are the sets $\{1\}, \{2\}, \{3\}$ (corresponding to the vertices themselves), $\{1,2\}, \{1,3\}, \{2,3\}$ (corresponding to the edges), and $\{1,2,3\}$ (the interior of the triangle itself). We refer to the face $\{1, 2, \ldots, n\}$ as the *interior* of a simplex, and removing the interior gives us the *boundary* of a simplex.

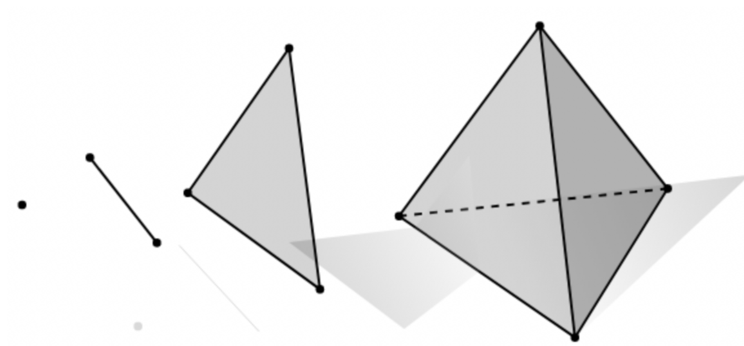


Figure 1. An example of 0, 1, 2, and 3-dim simplex

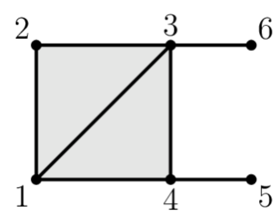
A *simplicial complex* is simply a collection of simplices. More formally, we have n vertices, along with a collection of faces between them which are part of different simplices. A simplicial complex must satisfy the following important property: if a face has vertex set S, then there is a face with vertex set T, for any $T \subset S$. We can also think of each face as a simplex of its own: the simplex with that face as its interior.

Now how does this have anything to do with Subset Takeaway? The answer is that we can embed a Subset takeaway position very naturally into a simplicial complex, which is hinted by the we have written the faces. In general, we have a simplicial complex with vertex set X, and each subset S that hasn't already been deleted is represented by a face with vertex set S. Now, why does this have to be a simplicial complex? Well, if we have some face with vertex set S, we know that all subsets of S must also be faces. It follows that S and its subsets correspond to a simplex, which we identify with S. Since this is true for all faces, it follows that this process will always yield a simplicial complex.

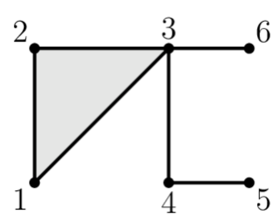
We can also consider playing Subset Takeaway on a simplicial complex. When we want to delete a set S, then we delete the face corresponding to S. We also delete all faces that contain the all the vertices of this face. This lets us rephrase Subset Takeaway in terms of simplices.

Definition 4 (Subset Takeaway on Simplicial Complices)**.** The impartial game Subset Takeaway is played on a simplicial complex on n vertices. The initial position is the boundary of a $(n-1)$-simplex. Allowable moves consist of deleting the interior of a simplex and all other simplices containing it as a face.

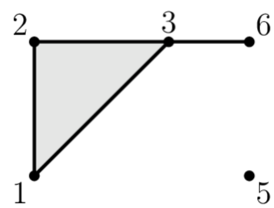
Example 2. Consider the following simplicial complex:



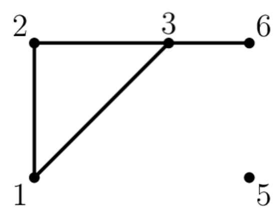
We use the notation $a_1 a_2 \cdots a_k$ to denote the set $\{a_1, \ldots, a_k\}$. Here, the allowable moves are all subsets of 123, 134, 45, and 36. Suppose Alice deletes the edge 14, giving us



If Bob now deletes the vertex 4 we get



Now, say, Alice erases the face 123. We get



If Bob responds by deleting vertex 3 and Alice counters by deleting edge 12, we are left with four isolated vertices $1, 2, 5, 6$. Since there are an even number of vertices, Alice wins.

Simplicial complexes are a construct that originate in topology, and using this new formulation, we can use the tools of topology to attack this problem.

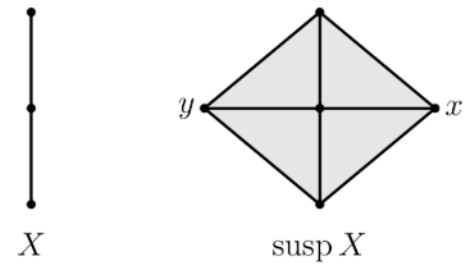
Christensen and Tilford [2] used this method to identify some useful strategies and prove some results using a so-called *reduction technique*. The gist is that we can apply certain constructions to a simplicial complex to simplify it, using the concept of a *binary star*, which is a pair of vertices of a simplicial complex that satisfy some nice properties. It turns out that removing these vertices will actually preserve the win/loss value of the game.

Using binary stars, they were able to verify that $n = 5$ was a second player win in just a few lines, compared to the large amount of manual labor required to do this by brute force. They also used binary stars to simplify the $n = 6$ case and then used a computer to prove that it too, was a second player win. We now describe this technique.

2.1. The Binary Star Reduction Method

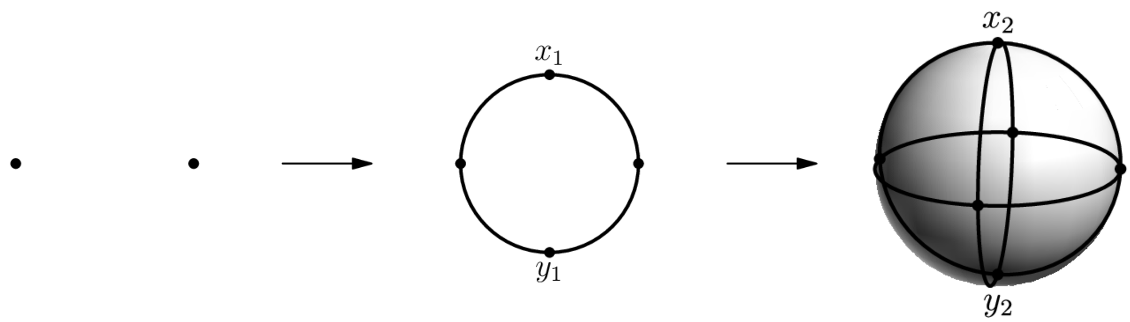
Let $A = \{1, 2, \ldots, n\}$, and consider a simplicial complex X on A. We will identify simplicial complices by the set of subsets of A corresponding to the faces. Now, we can define the *suspension* of X which is obtained by first adding two new vertices x, y to X. Then, for each face $a \in X$, we add the new faces $a \cup \{x\}$ and $a \cup \{y\}$. It turns out that X and $\operatorname{susp} X$ always have the same Grundy value; the proof follows from the more general case below. Using these, we can reduce more complicated positions to simpler ones: if we ever end up a position of the form $\operatorname{susp} X$, we can replace it with X.

Example 3. Consider the simplicial complex X as shown on the left below. Its suspension is shown on the right. So if we ever get $\operatorname{susp} X$ as a position, we can replace it with the much simpler position of X.



Using this reduction method, we might hope that we can use this to turn a starting position for n vertices in Subset-Takeaway into a starting position for some $m < n$ vertices. Then we would be able to recursively compute all the Grundy values.

It turns out that we can almost do this, but there is a subtle catch. If X is the boundary of a $(k+1)$-simplex, then X is topologically equivalent to a "k-sphere". The suspension of a k-sphere is a $(k+1)$-sphere, as shown below for $k=1$ and $k=2$.



Hence $\operatorname{susp} X$ is *topologically equivalent* to a $(k+1)$-sphere, which is topologically equivalent to the boundary of $(k+2)$-simplex. For example, if X is the boundary of a 1-simplex (which is just two isolated vertices) as shown on the left above, then $\operatorname{susp} X$ is as shown in the middle, which is topologically equivalent to a triangle, the boundary of a 2-simplex. However, $\operatorname{susp} X$ and a triangle are not equivalent Subset Takeaway games; and in general topologically equivalent simplicial complices need not have the same value. Therefore, we cannot directly apply the suspension reduction technique. However, there is a more powerful generalization of this known as the binary star technique.

For a pair (x, y) of vertices in X, we call (x, y) a **binary star** if:

- there is no edge between x and y;
- A face a contains x if and only if there is a face b that is identical to a except with x replaced with y.

It is easy to see how this is a generalization of the suspension, as in $\operatorname{susp} X$, all sets must contain exactly one of x and y, while in a binary star we may allow sets not containing either.

Proposition 5. *Let X' be the simplicial complex obtained from X by deleting the binary star x, y. Then X' and X have the same win/loss values.*

Proof. Consider the player who wins in X'. Then we give the strategy for them to win in X. They will always ignore the vertices x, y and play the winning strategy in X', *unless* the other player picks a set a, where a contains x. Then in the next move, they play the corresponding set b which contains y. And if the other player plays a set b containing y, in the next move they play the corresponding set a containing x.

It is easy to see that this strategy will guarantee a win. This completes the proof. □

In fact, the same proof works if we add any game to X' and X, so this implies that their Grundy values are the same, too.

Now we can illustrate the power of this technique.

Theorem 6. *Let $n \geq 3$ suppose Subset Takeaway is a second player win for $1, 2, \ldots, n-1$. Then, if the first player makes a move of size $1, 2, n-2$, or $n-1$, then the second player wins.*

By a move of size k, we mean deleting a subset of size k.

Proof. First suppose Alice makes a move of size 2, which is an edge connecting two vertices x, y. Then the resulting position is a binary star for x, y so by applying binary star reduction, this is equivalent to the position obtained by deleting x, y, which is a filled-in $(n-3)$-simplex. Then Bob can take the interior of this simplex, resulting in the position which is just Subset Takeaway for $n-2$. By the hypothesis for $n-2$, this is a second-player win.

Now suppose Alice makes a move of size 1, which consists of deleting a vertex x. This results in a filled in $(n-2)$-simplex, so Bob can remove the interior to get Subset Takeaway for $n-1$, which is a second player win.

Next suppose Alice makes a move of size $n-2$ so let x, y be vertices not in this move. Then Bob can delete x, y. By binary star reduction on x, y, the resulting position is equivalent to Subset Takeaway for $n-2$, so Bob wins.

Finally suppose Alice makes a move of size $n-1$, so let x be the vertex not in this move. Then Bob can delete x, and the resulting position is the same as in the second case, so Bob wins. □

By Observation 2, we know Subset Takeaway is a second player win for $n = 1, 2, 3$. Then, since moves of size $1, 2, n-2, n-3$ are the only possible moves for $n \leq 5$, from Proposition 3, it immediately follows that Subset Takeaway is a second player win for $n \leq 5$. Compared with the brute force method of determining the outcome, this is *much* easier.

Also, for $n = 6$, Proposition 3 implies that if the first player has a winning move, it has to be of size 3. Using this observation (and the hypothesis that playing the complementary set is a winning first move for Bob), Christensen and Tilford used a computer to prove that Subset Takeaway is a second player win for $n = 6$, proving Theorem 2.

Unfortunately, Brouwer and Christensen [1] used a computer program to show that $n = 7$ was in fact a first player win, thus disproving Gale's conjecture. As of now, there are no general conjectures on what the outcome should be for general n.

3 Variations and Generalizations

Subset Takeaway can be generalized to an arbitrary poset P, where if we take an element $x \in P$, then all $y \in P$ with $y \geq x$ are also removed. This game has been analyzed on many different posets, which are represented by lattices. For example, many cases have been solved if all subsets have no more than 2 elements, in which case the simplicial complex is just a graph, and the players are just deleting edges or vertices.

In [3], Khandhawit and Ye analyze Subset Takeaway on graphs. We state some of their results here.

Their main idea is a process called "involution reduction", which generalizes the binary star technique mentioned above.

Definition 7. For a simplicial complex X, suppose we have a function $\tau : X \to X$ satisfying:

(1) Restricted to the vertices of X, τ is a permutation such that $\tau^2(x) = x$ for any vertex x, and
(2) $a = \{a_1, \ldots, a_k\} \in X$ if and only if $\tau(a) = \{\tau(a_1), \ldots, \tau(a_k)\} \in X$.

Then we call τ an *involution.*

Their result is as follows.

Theorem 8. *Let X be a simplicial complex, and let τ be an involution on X. Also suppose that the fixed point set X^τ of τ is also a simplicial complex, so that there is no edge whose vertices are switched by τ. Then X and X^τ have the same Grundy values.*

If x, y is a binary star, then

$$\tau(z) = \begin{cases} y & \text{if } z = x, \\ x & \text{if } z = y, \\ z & \text{otherwise.} \end{cases}$$

is easily seen to be an involution satisfying the conditions of Theorem 5. Thus, Khandhawit and Ye's result generalizes the binary star reduction technique.

Next, Khandhawit and Ye used Theorem 5 to analyze Subset Takeaway on special types of graphs. We state their results here.

First they looked at complete n-partite graphs. To prove the following theorem, one does not need to use anything stronger than binary star reduction.

Theorem 9. *The Grundy value of $K_{a_1 a_2 \ldots a_n}$, the complete n-partite graph with components of size $a_1, \ldots, a_n$ is equal to $p \pmod 3$, where p is the number of odd terms among $a_1, \ldots, a_n$.*

Proof. We first look at the special case of the complete graph K_n, which happens when $a_i = 1$.

Lemma 10. *The Grundy value of the complete graph K_n is $n \pmod 3$.*

Proof. We prove this by induction on n; the base cases $n = 0, 1, 2$ are easily checked. Now suppose the statement is true for $0, 1, \ldots, n-1$. The first player has two moves in K_n, one of which is deleting a single vertex and the other being deleting a single edge. Taking a vertex results in K_{n-1}, which is equivalent to $*(n-1 \pmod 3)$ by the inductive hypothesis. Taking an edge gives us two vertices A, B which are not connected to each other, but connected to all other vertices. So they form a binary star, and thus we can delete them without changing the Grundy value. This gives us K_{n-2}, which is equivalent to $*(n-2 \pmod 3)$. Thus, K_n the Grundy value of K_n is

$$\text{mex}(n-1 \pmod 3, n-2 \pmod 3) = (n \pmod 3).$$

This proves the lemma. □

Now consider the general case, and suppose $a_i \geq 2$. Pick two vertices x, y in the i-th component, and note that they form a binary star as they are not connected. Then we can delete them, giving the new graph

$$K_{a_1 a_2 \cdots (a_i - 2) \cdots a_n}.$$

We can repeat this until all the a_i's are 0 or 1. Then since there are p of the a_i that become 1, it is simply the complete graph K_p. By the lemma, we are done. □

Next, they looked at general bipartite graphs. The classification has a rather nice form, depending only on the parity of the number of edges and vertices of the graph.

Theorem 11. *The Grundy value of a bipartite graph G is given by the following table, with $v(G)$ denoting the number of vertices and $e(G)$ denoting the number of edges:*

$v(g) \setminus e(G)$	even	odd
even	0	2
odd	1	3

They also analyzed odd-cycle pseudotrees, but the results are more complicated so we will not state them here.

References

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[2] J.D. Christensen and M. Tilford (2002). *David Gale's Subset Take-Away Game.* Amer. Math. Monthly **104** 762-766 [arXiv]

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