Universality and KONANE

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Abstract

When Conway introduced his theory of games, he used induction on Birthdays to construct all games. When combinatorial games are analyzed, we see that certain game values go with specific rulesets. This paper is an exposition of the results described in the paper by Carvalho and Santos [AC17], proving that the ruleset KONANE is the habitat for all short game values.

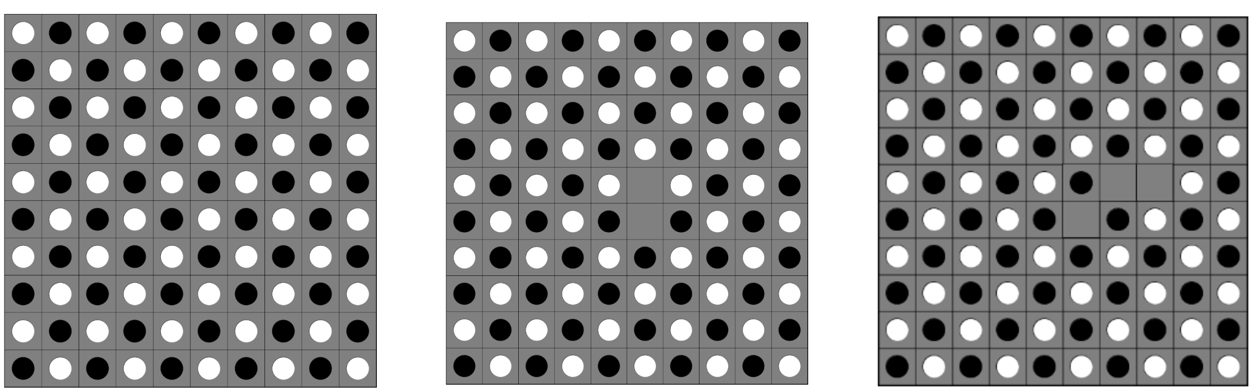
1 Introduction

1.1 Hakuna Matata! The game of Konane.

In this paper, we will study KONANE and its universal properties. Let us first see how to play KONANE.

Definition 1 (Rules of KONANE)**.** The game of KONANE starts with a checkered board filled with black (left) and white (right) stones in such a way that no two stones of the same color occupy adjacent squares. In the opening, two adjacent pieces of the board are removed; if left starts first, she removes a black piece and right removes a white piece next to the removed black piece, and vice versa. After this, a player moves by taking one of eir stones and jumping orthogonally over an opposing stone into an empty square. The stone that is jumped over is removed. A player can make multiple jumps in the same direction, and you do not have to make multiple jumps. The winner is the player who makes the last move. In this paper, we play with a generalized version of KONANE where two stones of the same color can occupy adjacent cells.

Here are some basic positions of KONANE.



(a) Start of KONANE game. (b) Removing adjacent pieces. (c) Black stone captures white stone.

Figure 1: Two stages of a KONANE game.

1.2 Game Values and Universality

Let us recall how to construct combinatorial games. We start by constructing the game 0, which is defined as $0 = \{|\}$. We say that the *birthday* of 0 is 0, or that 0 was born on day 0. To construct the games born on day 1, we take games whose options have birthdays 0 or less (and exclude games born on previous days). We can continue this for all integers.

Let's say we only want to consider games that can end. We can call them *short* games. We can observe that *short* games are the games that have birthdays less than ω.

Definition 2 (Short Conway Group)**.** We say that the *short Conway group* is the subgroup of the proper class of combinatorial games containing all short games, closed under the disjunctive sum.

As we study different rulesets of combinatorial games, we notice that certain rulesets only take up certain game values. Red-Blue HACKENBUSH games take on numbers as values, NIM games take on nimbers as values, and DOMINEERING has switches, numbers, and nimbers. We will introduce the notion of a habitat now.

Definition 3. Let $\mathscr{S}$ be a set of game values. We say that a ruleset R is a *habitat* for $\mathscr{S}$ if every game value in $\mathscr{S}$ arises as the game value of some game with ruleset R.

Using this definition, we can see that HACKENBUSH is a ruleset for the numbers. Essentially, a rule set R is a habitat for a set of values $\mathscr{S}$ if we can form all values in $\mathscr{S}$ in games of R. A ruleset is *universal* if its habitat is the short Conway group.

A natural question to ask is if there even exists a ruleset R that is the habitat of the short Conway group. That is, if there exists a *universal* ruleset. The answer to this question is yes! KONANE is a habitat for all short game values. The goal of this expository paper is to show that KONANE is universal.

Theorem 4 (Carvalho-Santos)**.** KONANE is universal.

We might be wondering why KONANE was suspected to be universal in the first place. Let us see how we can construct some numbers in KONANE. We have game values n in KONANE by taking a $2n + 1$ by 1 board and letting the top most square be black while alternate squares are white. For example, look at Figure 3a. Similarly, we would invert the colors for $-n$. We also observe that we have the game value $\frac{1}{2} = \{0 \mid 1\}$ in KONANE as shown in Figure 3b. We can continue building KONANE games with values $\frac{1}{2^n}$; see Figure 4.

Furthermore, let us look at nimbers that appear in KONANE. Santos proves in [CPDS08] that all nimbers appear in KONANE. This is through an iterative algorithm where we build $*n$ from $*(n-1)$. See Figure 2 for an example.

2 Building our Toolbox: Some Useful Konane Constructions

First, we will prove two preliminary lemmas that will help us prove the main theorem. The two constructions that will help us to prove Theorem 4 are the *rubber bands* and *taps*.

Definition 5 (Rubber Band)**.** The n^{th} *rubber band* P_n is a $(2n + 3) \times 5$ rectangle (indexed starting from 0), with black stones on the cells

$$(1, 2), (2, 2), (1, 4), (2, 4), \ldots, (1, 2n), (2, 2n),$$

and white stones starting on cells

$$(1, 0), (1, 2n + 2), (1, 1), (1, 3), \ldots, (1, 2n + 1),$$

$$(3, 2), (4, 2), (3, 4), (4, 4), \ldots, (3, 2n), (4, 2n).$$

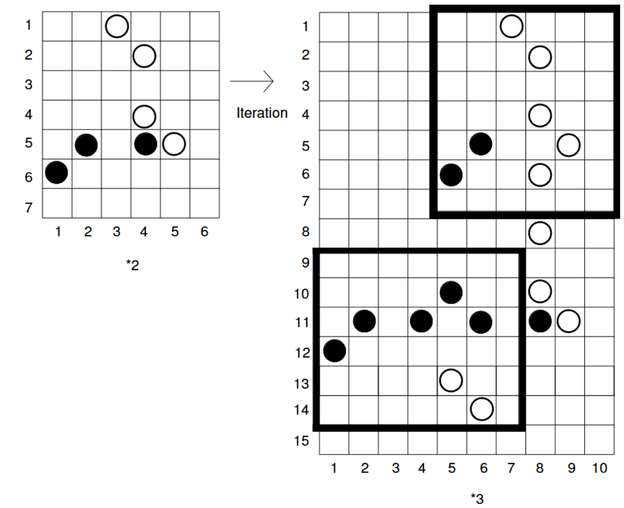
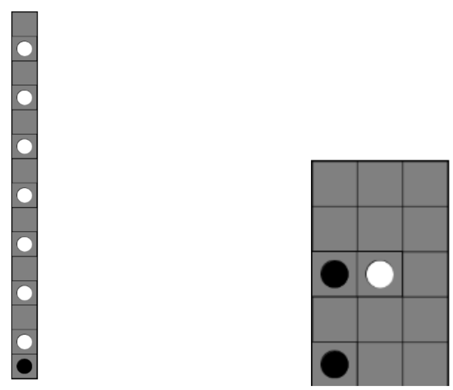


Figure 2: Building $*3$ from $*2$ iteratively.



(a) The number 7 in KONANE (b) The number $\frac{1}{2}$ in KONANE.

Figure 3: Two numbers in KONANE.

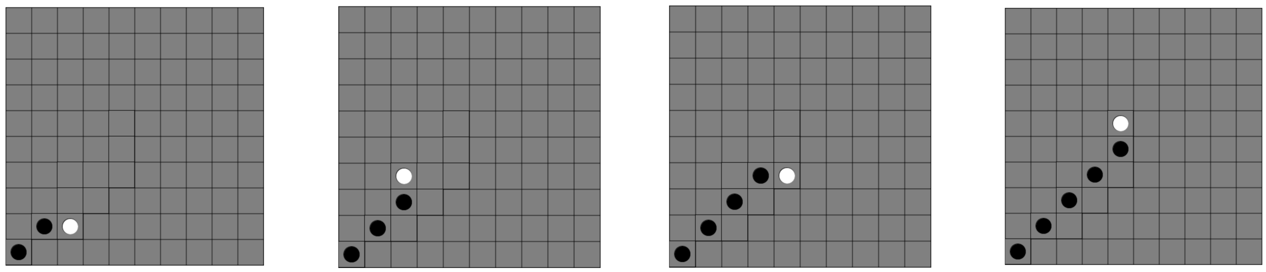
Lemma 6 (Values of Rubber Band)**.** Let $a(k)$ denote the number of black stones above row $2k+1$, while $b(k)$ denotes the number of black stones below row $2k+1$. Then we have the following values:

1. P_n has value 0.
2. $P_2 \setminus \{(1,3)\}$ has value -1.
3. $P_n \setminus \{(1,2k+1)\}$ has value $-\max(a(k), b(k))$ except for $P_2 \setminus \{(1,3)\}$.

Proof. The first two statements are easy to compute. The last statement is intuitive once you play a game of $P_n \setminus \{(1,2k+1)\}$.

1. This is easy to prove. Left and Right cannot make any moves, so $P_n \in \mathcal{P} \implies P_n = 0$.
2. The game tree for $P_2 \setminus \{(1,3)\}$ is shown in Figure 6. Thus its value is

$$\{\mid \{\{\mid 0\} \mid 0\}\} = \{\mid \{-1 \mid 0\}\} = \left\{\mid -\frac{1}{2}\right\} = -1.$$



(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

Figure 4: Numbers in the form of $\frac{1}{2^n}$ can be constructed as seen above.

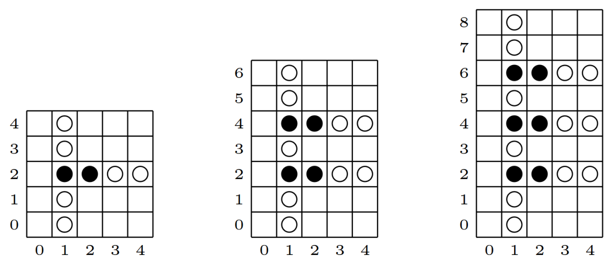


Figure 5: The rubber bands P_1, P_2, and P_3.

3. We can handle the remaining cases for $n \in \{1, 2\}$ by drawing similar game trees, so suppose that $n \geq 3$. Right has two options: he can either move the stone at $(1, 2k+3)$ to $(1, 2k+1)$, capturing the black stone at $(1, 2k+2)$, or he can move the stone at $(1, 2k-1)$ to $(1, 2k+1)$, capturing the stone at $(1, 2k)$.

   If he moves the stone from $(1, 2k+3)$, then Left to move can make one move, moving the stone from $(1, 2k)$ to $(1, 2k+2)$. Right's best move is then to make a double capture from $(1, 2k+5)$ to $(1, 2k+1)$. Right then has $a(k) - 1$ free moves, so that means that after Left's move, the resulting position has value $-a(k)$.

   On the other hand, after Right's initial move, if Right moves again, then he can move the stone at $(3, 2k+2)$ to $(1, 2k+2)$, yielding a position where Left has no more moves and Right has an extra $a(k) - 2$ moves. Thus the resulting position has value $2 - a(k)$, so the position after Right's first move has value $\{-a(k) \mid 2 - a(k)\} = 1 - a(k)$.

   Similarly, if Right's first move is to capture up, from $(1, 2k-1)$ to $(1, 2k+1)$, then the resulting position has value $1 - b(k)$. So the position $P_n \setminus \{(1, 2k+1)\}$ has value

   $$\{\mid 1 - a(k), 1 - b(k)\} = -\max(a(k), b(k)),$$

   as desired.

   Remark 7. The lemma basically says if you remove a white stone in the long part of the rubber band, Right can go up or down to score the maximum number of black stones possible.

□

Definition 8 (Taps)**.** The nth taps position T_n is a $7 \times (2n+3)$ rectangle, with black stones on the cells

$$(2n+2, 2), (2n+2, 3), (2, 4), (3, 2), (4, 4), (5, 2), \ldots, (2n, 4), (2n+1, 2),$$

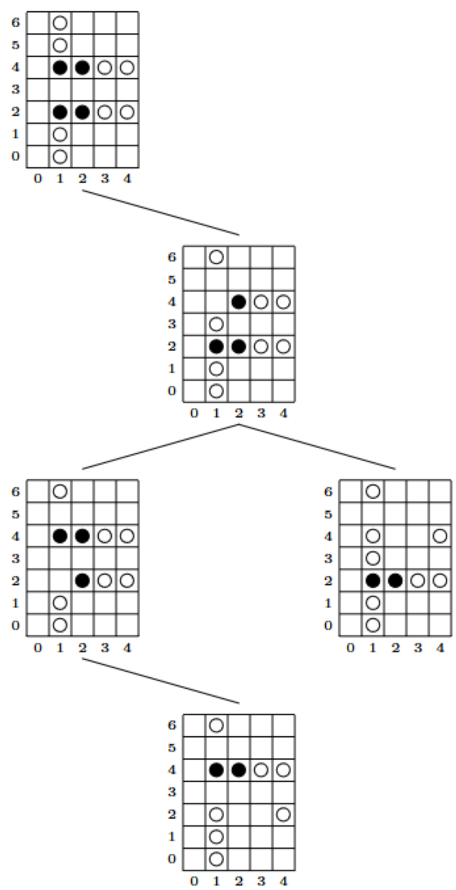


Figure 6: The game tree for $P_2 \setminus \{(1,3)\}$

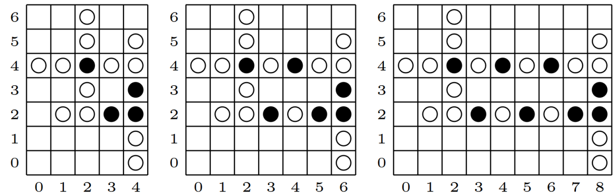


Figure 7: The taps positions T_1, T_2, and T_3.

and white stones on the cells

$$(0,4),(1,2),(1,4),(1,4),(2,2),(2,3),(2,5),(2,6),$$

the cells

$$(2n+2,0),(2n+2,1),(2n+2,4),(2n+2,5),$$

and the cells

$$(3,4),(4,2),(5,4),(6,2),\ldots,(2n,2),(2n+1,4).$$

Lemma 9 (Values of Taps)**.** The following statements hold for all T_n.

1. T_n has value 0.
2. $T_n \setminus \{(1,2)\}$ has value $n+1$.
3. $T_n \setminus \{(1,4)\}$ has value $-n$.
4. $T_n \setminus \{(1,2),(1,4)\}$ has value 0.

Proof. We will prove each statement by just computing.

1. Neither side has any moves available.
2. First note that row 4 with the stone at $(4,2)$ removed has value. Thus, Left can take the n white stones on row 2 together with the white stone at $(2,3)$ before reaching a 0 position, without Right having any moves available, so the value is $n+1$.
3. Right can take the n black stones on row 4, and Left has no moves.
4. Left has n moves on row 2, and Right has n moves on row 4, so the value is 0.

□

3 The Universality of Konane

Theorem 10 (Carvalho-Santos)**.** Konane is universal.

Proof. We first check that the games born on days 0 and 1 appear in konane, by showing positions achieving these values; see Figure 8. Now, note that we can add rubber bands to a position without changing the value, as shown in Figure 9, with Right being able to capture down and Left being able to capture up.

Now, let's investigate the options available to the players. Let's look at just the left options, for the right options can be handled similarly. Initially, the only black stone that can move is the one at the focal point,

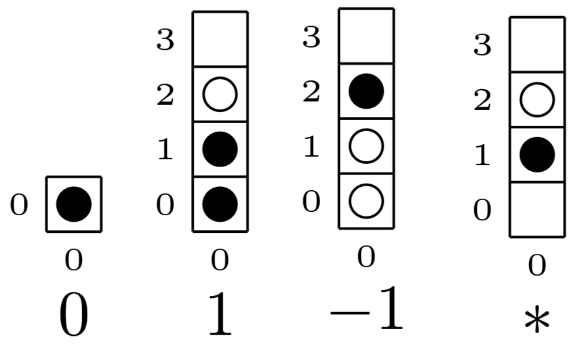


Figure 8: Games of birthdays 0 and 1

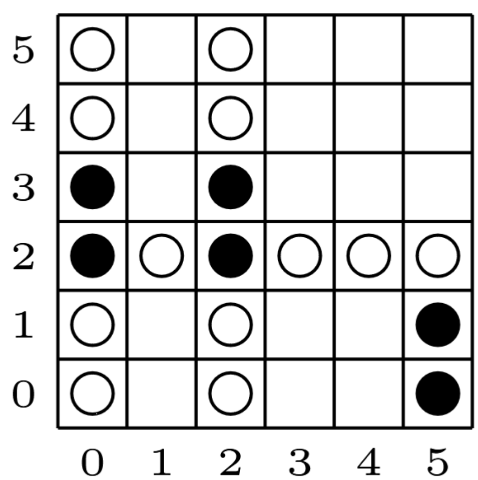


Figure 9: We combined the game of value 1 with a rubber band, and the value didn't change.

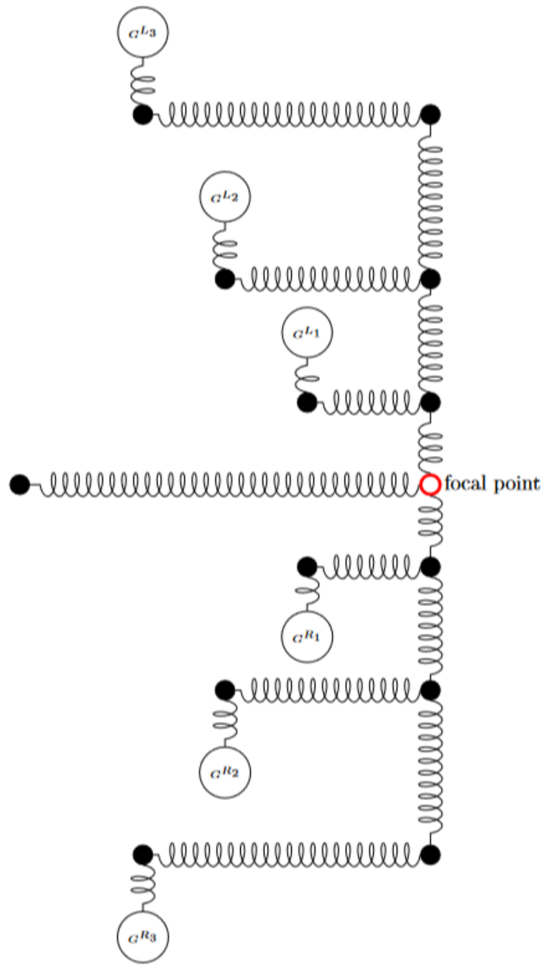


Figure 10: The entire schematic for constructing G through KONANE [RS]

and it can capture up along its column. There are two kinds of moves available: either to one of the black dots, or to somewhere in the middle of one of the rubber bands.

We first show that, if the rubber bands are chosen to be sufficiently long, then a move to one of them reverses out. Let us suppose that all the options of G have birthday at most n. Then we let all the rubber bands from the focal point have length at least $2n+2$. If Left moves to the middle of such a rubber band, as shown in Figure 11, then Right has a move to at least $-n-1$. Since G has birthday at most $n+1$, we have $-n-1 \leq G$, so this left option is reversible, and we can replace it with the left options of $-n-1$ (i.e., the empty set).

Now we have to consider the other type of move, to one of the black dots, say the one leading to G_{L1}. From here, Left has options to move further in the vertical direction, thus stopping in the middle of some other rubber band (which, as we have seen, is a dominated option and can thus be omitted from the list of options), or else at one of the remaining black dots. But we also give Left an option of moving to a large positive number, so that Right is forced to respond to the threat rather than move elsewhere. If s and t are large enough, then we may replace the option G_{L_1} with the new option $\{s \mid \{G_{L_1} \mid -t\}\}$ without changing the value of the game.

The setup in konane is illustrated in Figure 12. Suppose that it is Left to move, and she is to move a black stone along column 21. If she moves to $(21,4)$, then Right moves the stone at $(22,4)$ to capture the black stone at $(21,4)$, and thus ends up with at least $-n$. Thus this move is reversible to the empty set and

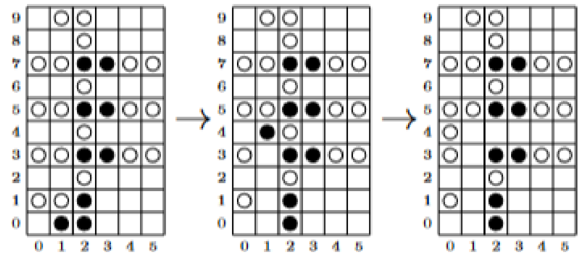


Figure 11: Moving to the middle of a rubber band is a reversible option.

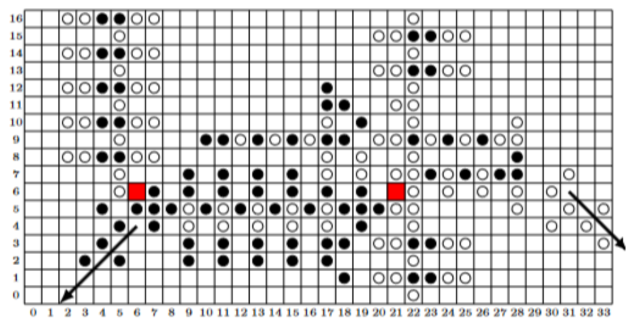


Figure 12: Here is how a turning point looks like.

can thus be removed. If Left moves to $(21, 8)$, then Right moves the stone at $(21, 9)$ to capture the stone at $(21, 8)$, scoring at least $-n$, because we have reached a taps position. Thus this left option is reversible to the empty set and can thus be removed. If Left moves to $(21, 10)$, then Right moves the stone at $(22, 10)$ to $(18, 10)$. Right will then move from $(18, 10)$ to $(18, 8)$, scoring at least $-n$. Thus this left option reverses to the empty set and can thus be removed.

Finally, we address the critical move for Left, which is to $(21, 6)$, marked in red. With this move, she creates a threat of capturing to $(31, 6)$, with a score of at least n. If Right responds by moving from $(22, 6)$ to $(20, 6)$, then Left responds by moving from $(20, 5)$ to $(20, 7)$. Then, the move from $(20, 7)$ to $(16, 7)$ scores at least n points. Thus the right option to $(20, 6)$ reverses to the empty set and can be removed from the list of options. Similarly, if Right moves from $(22, 6)$ to $(18, 6)$, then Left responds by moving from $(18, 5)$ to $(18, 7)$, scoring at least n points. Thus the right option to $(18, 6)$ reverses to the empty set and can thus be removed.

The critical response for Right, then, is to move from $(22, 6)$ to $(6, 6)$. This creates a threat of moving from $(6, 6)$ to $(6, 4)$, scoring at least $-n$. Left must respond to this threat by moving up from $(6, 5)$ along column 6, reaching the option of G_{L_1}. Thus the left option in the original position of moving to $(21, 6)$ has value $\{s \mid \{G_{L_1} \mid -t\}\}$ after bypassing some reversible options, and for suitably large values of s and t, which can be arranged by making the sequences shown in the arrows sufficiently long, this as an option in G can be replaced with G_{L_1}. This completes the proof. □

4 Examples

We will see the game $G = \{0, \uparrow *, \pm 1 \mid\mid -1, \{* \mid -1\}\}$; the KONANE game is shown in Figure 13. If we remove the white stone from the cell $(37, 29)$ and place a black stone in the cell $(38, 29)$—a scenario resulting from the capture of the white stones in row 29—the resulting game is $\{0, \uparrow *, \pm 1 \mid\mid -1, \{* \mid -1\}\}$. Note that the colored rectangles represent the options from previous days.

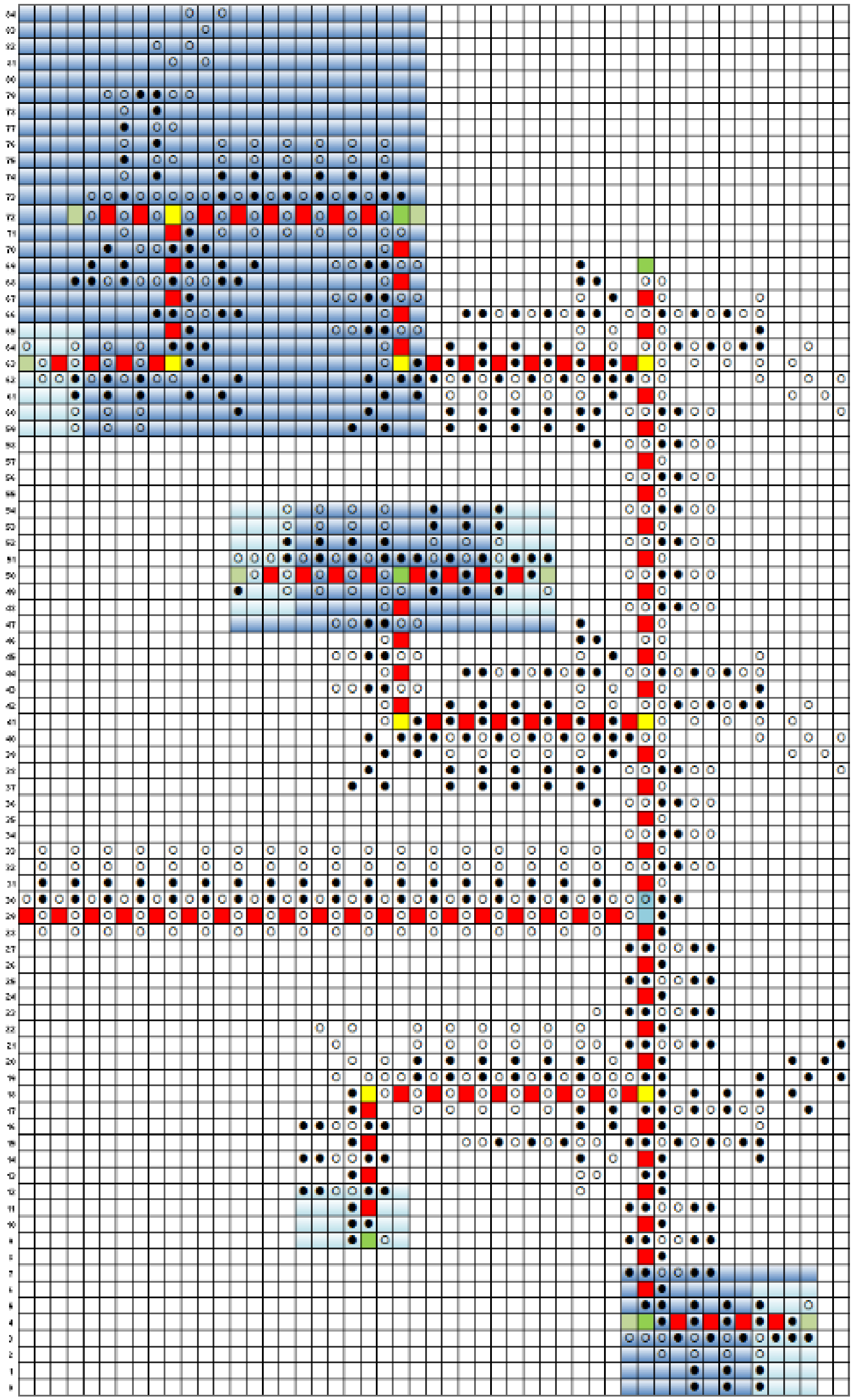


Figure 13: Terminal position for future constructions.

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