Classical Impartial Games

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November 2024

1 Introduction

Impartial games are an important category of combinatorial games. In these games, the same moves are available to both players, and the first player who cannot make a move loses. They can be analyzed with Sprague-Grundy theory, which allows impartial games to be easily reduced to nimbers. Nimbers are the games $0, *, *2, \ldots,$ each of which has moves to every previous nimber. Some of the most famous classical impartial games reduced to nimbers are FIBONACCI NIM and WYTHOFF'S GAME. These are both modifications of NIM, which is the most important impartial game. FIBONACCI NIM is a game where players can remove a certain number of coins from a pile, and the number of coins that can be removed is at most twice the number of coins removed on the last turn. In WYTHOFF'S GAME, there are two piles, and players have a choice between removing a number of stones from one pile or removing the same number of stones from both piles. In this paper, we will find the winning strategies for both WYTHOFF'S GAME and FIBONACCI NIM.

2 Preliminaries

Definition 2.1. Positions in WYTHOFF'S GAME are represented by a pair of nonnegative integers (a, b), representing the number of coins in each pile. Valid moves from (a, b) are:

1. $(a - n, b)$ where $0 < n \leq a$
2. $(a, b - n)$ where $0 < n \leq b$
3. $(a - n, b - n)$ where $0 < n \leq \min\{a, b\}$

Example 2.1. $(1, 1)$ would be a $\mathcal{N}$ position since the first player can move to $(0, 0)$, but $(1, 2)$ would be a $\mathcal{P}$ position since all valid moves are to an $\mathcal{N}$ position.

Definition 2.2. Positions in FIBONACCI NIM are represented by a pair of nonnegative integers (n, r). n represents the number of coins in the pile, and r represents the maximum number of coins that can be taken away on the next move. Valid moves from (n, r) are:

1. $(n-k, 2k)$ where $0 < k \leq \min\{n, r\}$

The starting positions of FIBONACCI NIM we are interested in are of the form $(n, n-1)$. This gives the first player the maximal number of choices without giving them an instant win by removing all the coins.

Example 2.2. If the starting position is $(4, 3)$, and the first player removes 1 coin, then the resulting position is $(3, 2)$. At this point the second player can remove either 1 or 2 coins. Either choice allows the first player to move to a $(0, r)$ position, which causes the second player to lose. Therefore $(4, 3)$ is a $\mathcal{N}$ position.

To find a winning strategy for impartial games, we need to classify which positions are $\mathcal{P}$ and which are $\mathcal{N}$. The winning strategy from a $\mathcal{N}$ position is to move to a $\mathcal{P}$ position. $\mathcal{P}$ positions are a loss for the first player, so there is no winning strategy. To find the $\mathcal{P}$ and $\mathcal{N}$ positions, we use the partition theorem:

Theorem 2.1. *Let $\mathscr{L}$ be a set of impartial games closed on subpositions. Suppose $\mathscr{P}$ and $\mathscr{N}$ partition $\mathscr{L}$ such that every move from a game $G \in \mathscr{P}$ is to a game in $\mathscr{N}$ and for every $G \in \mathscr{N}$, there is a move to a game in $\mathscr{P}$. Then $\mathscr{P} = \mathcal{P} \cap \mathscr{L}$ and $\mathscr{N} = \mathcal{N} \cap \mathscr{L}$.*

To classify the positions of FIBONACCI NIM, we need Zeckendorf's theorem:

Theorem 2.2. *Let F_n denote the n^{th} Fibonacci number, and let k be any non-negative integer. Then there exists a unique sequence $\{c_i\}_1^r$ where $c_i \geq 2$ and $c_i \geq c_{i+1} + 2$ for all i, such that*

$$\sum_{i=1}^{r} F_{c_i} = k$$

For WYTHOFF'S GAME, we will need Rayleigh's theorem:

Theorem 2.3. *Let r and s be two positive irrational numbers with $\frac{1}{r} + \frac{1}{s} = 1$. Then every positive integer n can be uniquely represented as $\lfloor rk \rfloor$ or $\lfloor sk \rfloor$ for some positive integer k.*

Proof sketch: We want to prove that $\lfloor rk \rfloor \neq \lfloor sm \rfloor$ for integers k, m, to ensure uniqueness. If $j = \lfloor rk \rfloor \neq \lfloor sm \rfloor$, this gives the inequalities $j < rk < j+1$ and $j < sm < j+1$, since r and s are irrational. Dividing the first by r and the second by s, then adding the two inequalities, results in $j < k + m < j + 1$, a contradiction since $k + m$ is an integer. A similar contradiction can be obtained to prove existence.

Apart from analyzing which positions are $\mathcal{P}$ or $\mathcal{N}$, we may also want to know the Grundy values of positions of these games, which describe how they affect sums of impartial games. In Figure 1, we show a chart of the Grundy values of FIBONACCI NIM for $0 \leq n < 500$ and $0 \leq r < 500$. In Figure 2, we do the same for WYTHOFF'S GAME. General formulas for the Grundy values of either game are not yet known. Larsson and Rubinstein-Salzedo [2016]

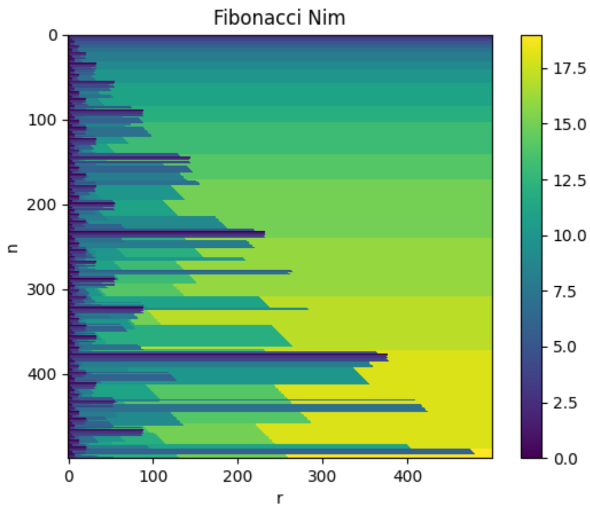

Figure 1: Grundy values of positions in Fibonacci nim

3 Proof

First, let $\mathcal{L}$ be the set of all positions (n, r) of Fibonacci nim. Let N be a positive integer. By Zeckendorf's theorem there is a nonempty sequence $\{c_i\}_1^k$ which are the indices of Fibonacci numbers that sum to N. Let $g(N) = c_k$. Define

$$\mathcal{P} = \{(n, r) \colon n = 0 \vee r < g(n)\}$$

and

$$\mathcal{N} = \{(n, r) \colon n > 0 \wedge r \geq g(n)\}$$

Certainly these sets partition $\mathcal{L}$, so we must check the other requirements of the partition theorem. We have two cases to consider: (n, r) can be in $\mathcal{P}$ or $\mathcal{N}$.

- Suppose $(n, r) \in \mathcal{N}$, and let $\{c_i\}_1^k$ be the Zeckendorf representation of n. Then removing $g(n)$ coins is a valid move to $(n - g(n), 2g(n))$. If $n - g(n) = 0$, then this is in $\mathcal{P}$. Otherwise, $n - g(n)$ has Zeckendorf representation $\{c_i\}_1^{k-1}$, so

$$g(n - g(n)) = F_{c_{k-1}} \geq F_{c_k+2} = F_{c_k+1} + F_{c_k} > 2F_{c_k} = 2g(n)$$

  Thus $(n - g(n), 2g(n)) \in \mathcal{P}$, so we have found a move to a position in $\mathcal{P}$.

- Suppose $(n, r) \in \mathcal{P}$. If $n = 0$, then no moves are available, and we are done. Otherwise, the available moves remove m coins, where $m \leq r < g(n)$. Suppose $(n - m, 2m) \in \mathcal{P}$. Clearly $n - m > 0$, since $m < g(n) \leq n$. Therefore we have $2m < g(n - m)$. Let $\{c_i\}_1^k$ be the Zeckendorf

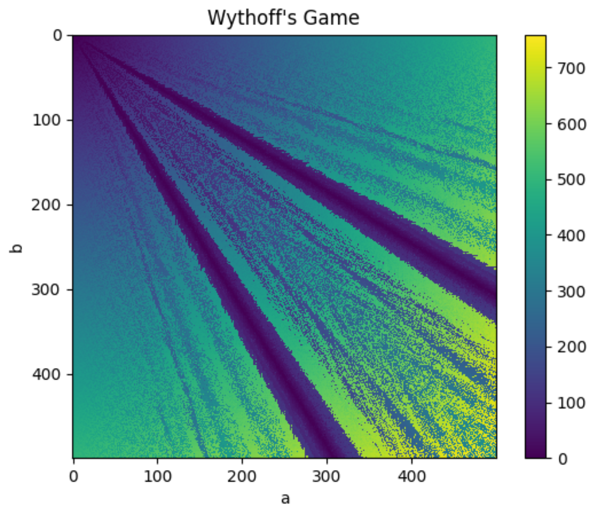

Figure 2: Grundy values of positions in Wythoff's game

representation of n. If $c_k = 2$, then $g(n) = 1$, so m would have to be 0, a contradiction. Thus $c_k > 2$. Now let l be maximal such that

$$F_{c_k - 2l} > m$$

Then we have

$$\begin{aligned} n - m &= \left(\sum_{i=1}^{k} F_{c_i}\right) - m = \left(\sum_{i=1}^{k-1} F_{c_i}\right) + F_{c_k} - m \\ &= \left(\sum_{i=1}^{k-1} F_{c_i}\right) + \sum_{j=1}^{l} F_{c_k - 2j + 1} + (F_{c_k - 2l} - m) \end{aligned}$$

Note that the two sums on the left have the necessary requirements to be a Zeckendorf representation. The term on the right is positive and less than F_{c_k-2l}, so its Zeckendorf representation can be appended to the two other sums to get the Zeckendorf representation of $n - m$. Thus we have $g(n-m) \le F_{c_k-2l-1}$. But since l is maximal, we know that $F_{c_k-2l-2} \le m$. Therefore $2m < g(n-m) \le F_{c_k-2l-1} = F_{c_k-2l-2} + F_{c_k-2l-3} \le 2F_{c_k-2l-2} \le 2m$, a contradiction. Hence $(n - m, 2m)$ must be in $\mathcal{N}$, so this requirement of the parittion theorem holds.

By the partition theorem, $\mathcal{P}$ and $\mathcal{N}$ are the $\mathcal{P}$ and $\mathcal{N}$ positions of $\mathcal{L}$ respectively. We are mainly interested in positions of the form $(n, n-1)$ where $n > 1$. If n is a Fibonacci number, then $g(n) = n$, so $(n, n-1) \in \mathcal{P}$. Otherwise, $g(n) < n$, so $n - 1 \ge g(n)$ and $(n, n-1) \in \mathcal{N}$. Therefore the first player can win if and only if n is not a Fibonacci number.

In the case of WYTHOFF'S GAME, the classification of positions is quite different. Since the positions (a, b) and (b, a) are equivalent, we only consider positions with $a \leq b$. In this case, we define

$$\mathcal{P} = \left\{ \left(\left\lfloor n \frac{1+\sqrt{5}}{2} \right\rfloor, \left\lfloor n \frac{3+\sqrt{5}}{2} \right\rfloor \right) : n \geq 0 \right\}$$

and $\mathcal{N}$ to be all other positions. We want to prove that all moves from $\mathcal{P}$ positions are $\mathcal{N}$ positions. Let $(a, b) \in \mathcal{P}$. If $a = 0$ and $b = 0$, then there are no moves, so we are done. Otherwise, by Rayleigh's theorem, when $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{3+\sqrt{5}}{2}$, every positive integer occurs exactly once as a pile size in a $\mathcal{P}$ position. Therefore removing coins from one pile will not give a $\mathcal{P}$ position. Also, every positive integer occurs exactly once as the difference between the sizes of the piles in a $\mathcal{P}$ position. Therefore removing the same number of coins from each pile will not give a $\mathcal{P}$ position. Now suppose $(a, b) \in \mathcal{N}$. If $a = 0$, then removing all coins from the second pile moves to a $\mathcal{P}$ position. Otherwise, suppose that $a = \left\lfloor n \frac{1+\sqrt{5}}{2} \right\rfloor$ for some positive n. If $b > \left\lfloor n \frac{3+\sqrt{5}}{2} \right\rfloor$, then moving the second pile to $\left\lfloor n \frac{3+\sqrt{5}}{2} \right\rfloor$ gives a $\mathcal{P}$ position. If $b < \left\lfloor n \frac{3+\sqrt{5}}{2} \right\rfloor$, we have $0 \leq b - a < n$. Therefore we can move to

$$\left(\left\lfloor (b-a) \frac{1+\sqrt{5}}{2} \right\rfloor, \left\lfloor (b-a) \frac{3+\sqrt{5}}{2} \right\rfloor \right)$$

which is a $\mathcal{P}$ position. Now suppose $a = \left\lfloor n \frac{3+\sqrt{5}}{2} \right\rfloor$ for some positive n. Then, since $b \geq a$, we can move the second pile to $\left\lfloor n \frac{1+\sqrt{5}}{2} \right\rfloor$, which is a $\mathcal{P}$ position. Thus $\mathcal{P}$ and $\mathcal{N}$ have the necessary properties for the partition theorem.

4 Conclusion

By finding the winning strategies for FIBONACCI NIM and WYTHOFF'S GAME, we gain insight into potential interesting extensions of the two games. An obvious extension for FIBONACCI NIM is to increase the bound on the next move to some other integer multiple, such as 3. This could lead to a generalized variant of the Zeckendorf representation. For WYTHOFF'S GAME, a possible extension would be to allow any number of piles. A valid move would be to remove n coins from a nonempty subset of piles. Also, the Grundy values of FIBONACCI NIM appear to have a fractal structure that could be analyzed with more work.

References

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