Application of Combinatorial Game Theory to Chess

Mihika Deshpande

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1 Introduction

Chess is a complex, combinatorial, game that dates back to the 6th century CE, distantly related to ancient board games like Chaturanga, the Sanskrit name for a battle formation. It involves two players, white and black, who move their respective chess pieces across the 8-by-8 board until one of them wins by trapping the other's king [Sol24]. Combinatorial game theory is a branch of mathematics that involves analyzing the optimal strategy and moves in a given combinatorial game. For convenience, White and Left will be interchangeable terms and Black and Right will be interchangeable terms.

Traditionally, chess is not a game analyzed by combinatorial game theory. One reason for this that the same position can be reached multiple times in a game, creating loops.

In addition, the disjunctive sum used to add combinatorial games breaks down in chess, since very few practical positions contain separate games that don't interact with each other. One example of a game in which disjunction sums works is in numbers where only pawns play in the game, like in Figure 1. This is because pawns don't have the freedom to move across the entire board, preventing games from interacting with each other [Elk99]. In contrast, Figure 2 shows how games with higher ranking pieces shows how high ranking pieces typically interact across the whole board, breaking down disjunctive sums.

For the following sections, we're are going to be creating positions, solely with pawns. This will aid us in evaluating real game positions with Kings.

2 Numbers

Since numbers build off of each other, we can start by creating positin for integers and then move to addressing dyadic rationals.

Integers

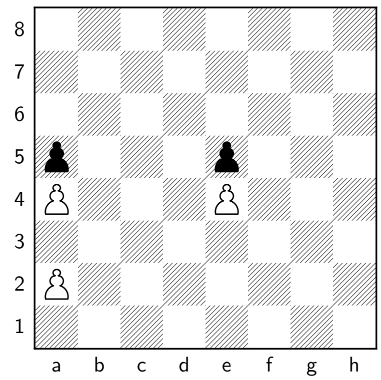

Figure 1: Sum of 0 game and 1 game.

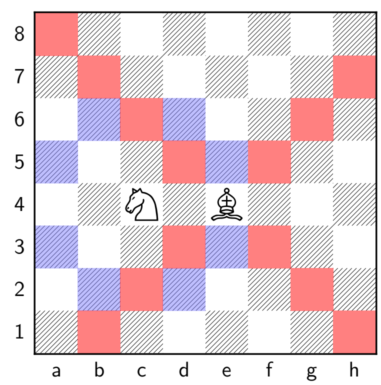

Figure 2: Options using the Knight (blue) or Bishop (red).

We can start by defining the 0 game in chess, and then adding pawns to create the numbers. The most obvious zero position is just an empty board, since neither Left nor Right has a move. However, this doesn't aid us in constructing other numbers, so we can define the zero position in Figure 3.

Since pawns only move forward one square, and there are no pieces nearby to capture the pawns, neither Left nor Right has a move, so this is a zero position. Any permutation of this position, in any location on the board will result in a zero game.

When we start adding pawns in positions where they can capture and interact with each other, we get positions similar to Figure 4.

In Figure 4, two additional pawns than Figure 3 are added [WYL10]. However,

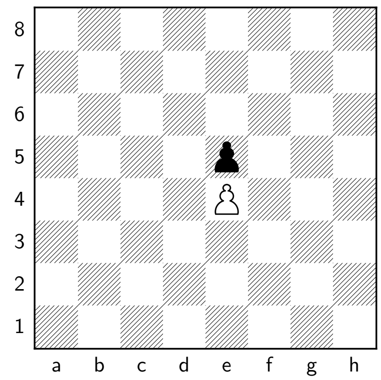

Figure 3: 0 game.

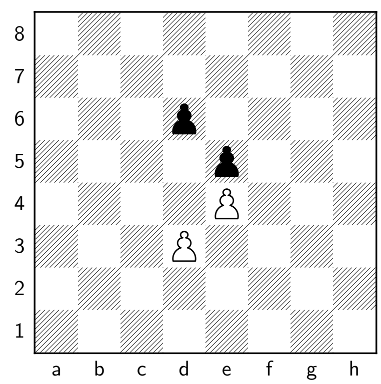

Figure 4: 0 game, without only blocked pawns.

Figure 4 is still a zero position, since if either the d3 or the d6 pawns move forward, the other player can capture the pawn and then continue forward to promote. This leads us to our first definition.

Definition 1. If a pawn *promotes*, meaning it reaches either the first rank (row) for white pawns or eigth rank for black pawns, the pawn turns into a high-ranking piece, winning that player the game.

If we add an additional pawn to Figure 3 in the same rank, we get the position in Figure 5.

In Figure 5, Left has one move to e3 and Right doesn't have any moves, so this is the one game. The other integer games can be created similarly, by giving one side tempos.

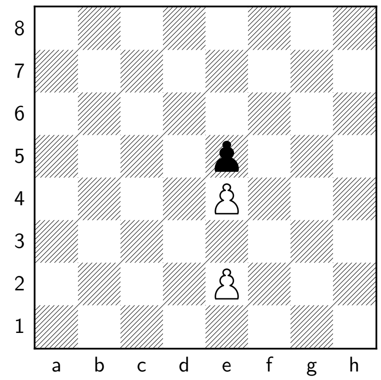

Figure 5: 1 game.

Definition 2. *Tempos* are additional moves the one side has in chess that can be used to force the other side to play. They function almost as a means for stalling in the game.

One thing to note is that since the chessboard is a finite size, there is a finite amount of integers that can be represented. Specifically, only -12 through 12 can be represented using variations of the position in Figure 5, while still technically maintaining a legal chess position. This position can be found in Figure 6.

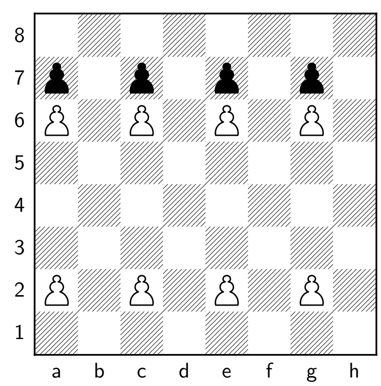

Figure 6: 12 game.

We have to leave a file between each 3 game, since otherwise the games interact with each other.

2.1 Dyadic Rationals

Since the pawn moves in chess rely on having an open square in front of a given pawn, if there are open spaces between a White and Black pawn, both pawns have the same options. If there is a space between two pawns of the same color, there is a move to an integer game one below the current game by moving the not blocked pawn forward. In order to create a dyadic, like $\{1|0\}$, one player has to have options that the other player does not have. This only happens in circumstances like the one shown in Figure 7, where the left side of the board is a zugzwang position, forming a zero position and black has an extra tempo in the h-pawn. [Elk99].

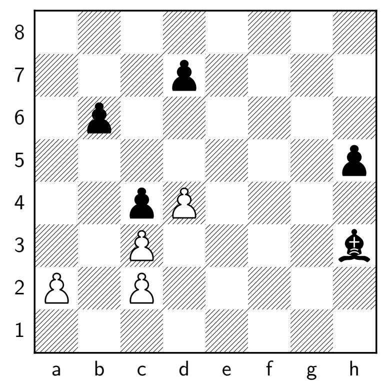

Figure 7: $\frac{1}{2}$ game.

Definition 3. *Bishops* can't move. At the beginning of this paper, we established that bishops are high-ranking pieces and aren't included in our analysis because they can move across the board. The exception for this is cases where several pieces on the board block the high-ranking piece's options, effectively turning it into a block that can't move.

For the purpose of creating a dyadic rational with simple positions, we are going to assume that the bishop cannot move in Figure 7. In real play, the kings and other pieces would assist in blocking the bishop and confining it to h3.

However, note that this doesn't work for creating all the dyadic rationals for a similar reason that disjunctive sums break down. Since in pawn games in the same rank, Left's options don't effect Right's options, if Right had the option to make a move, Left can't take that option away, rendering the dyadic impossible to create with one-rank pawn positions. With whole board positions like the one showed in Figure 7, the position takes up almost the whole board, severely limiting the fractions we can contruct.

3 Other games

Just by placing pawns in other one-rank positions such that they have access to the same move creates non-number positions. For example, the position in Figure 8 creates the *, or $\{0|0\}$ game.

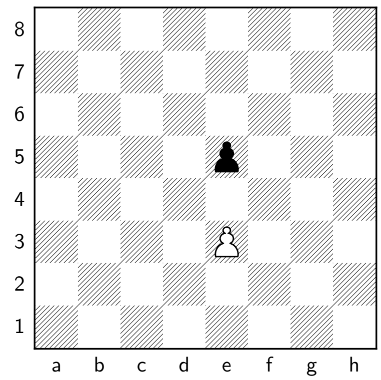

Figure 8: * game.

If both white and black pawns in the same file aren't deadlocked and there aren't pawns in the surrounding files, both pawns have access to the same moves, so a single space between the pawns gives us $\{0|0\}$ [WYL10].

Another property of pawns is being able to move two squares.

Definition 4. If a pawn is in it's original position on the board, the player has the choice of moving the pawn one or two squares forward in a single move.

Using this property, one side can have more moves than the other side with pawns still in a single file. This gives us the position in Figure 9 [WYL10].

In Figure 9, the white pawn can move either to e3 or e4, while the black pawn can only move to e4, so this forms the option set, $\{0|*\}$, which can be simplified to $\{0|*\}$, since white's * leads to black moving to 0, causing white to lose. $\downarrow$ can be formed by flipping the position to have the black pawn on the seventh rank and the white pawn on the fourth rank.

4 Trebuchet positions

In the previous sections, we showed some examples of number positions and positions for other games with only pawns. Since chess games require kings on the board, none of these positions are applicable to real chess positions. However, since Kings can move to any square around the their current square, adding kings would change the values of all of the positions we defined.

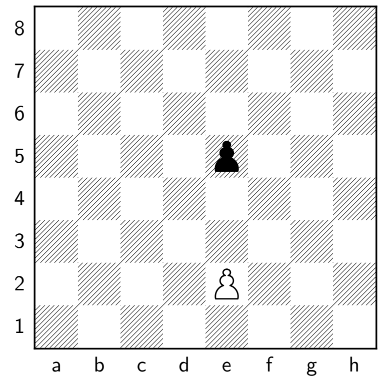

Figure 9: $\uparrow$ game.

We can fix this issue by creating trebuchet positions.

Definition 5. A *trebuchet* position is a position where neither player should move their King, because it is a bad move.

We can see such a position in Figure 10.

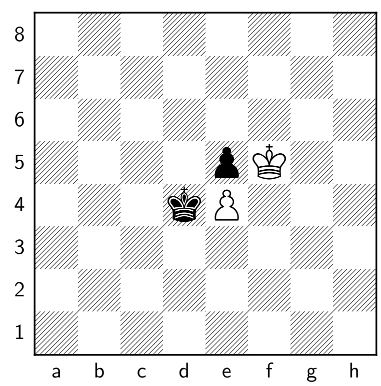

Figure 10: Trebuchet position.

In this position, if either player makes a move, their King is forced to move away from protecting their own pawn and the other player's King can capture the pawn and promote their own, winning the game.

This means that if we add Trebuchet positions to another game, we create plausible positions with the value of that game.

For example, the position in Figure 11 adds a trebuchet position to the 1 game, creating a plausible 1 game position.

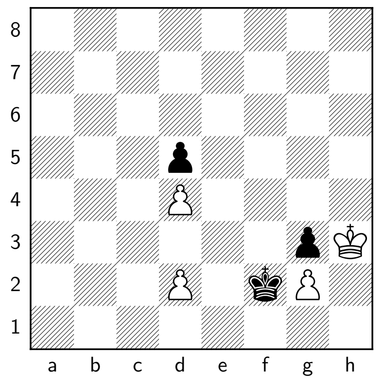

Figure 11: Trebuchet 1 game.

5 Conclusion

Chess is a complex game, and the variety of pieces, as well as the large amount of moves each high-ranking piece has, makes it difficult to evaluate all possible chess positions. However, when it comes to endgames with pawns in particular, it becomes much more feasible to analyze chess positions with combinatorial game theory.

We've already explored how to represent several positions with pawns, how Kings can be included in Trebuchet positions to not affect the value of pawn games, and how high-ranking pieces can be added to positions without affecting the value by blocking the piece in.

However, there are still potential, realistic advances to consider for analyzing chess games using combinatorial game theory. Is there a way to create non-dyadic rational fractions on a chess board [Elk99]. Is there a way to represent single high-ranking piece endgames, like the rook and king mate, and derive the winning strategy?

References

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