ON MAKER-BREAKER GAMES

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Abstract. Maker-Breaker games are positional games in which one player, Maker, has the objective of getting all the elements of some winning set in the game, while the other player, Breaker, has the objective of preventing that goal of Maker. In other words, Breaker would win if they obtain at least one element of every winning set. This paper provides a simple understanding and basic entry of Maker-Breaker games. Provided are a few easy-to-understand Maker-Breaker games, simple Maker-Breaker notation, and strategies that players can adopt to win Maker-Breaker games.

1. introduction

Positional games are combinatorial games that are played on hypergaphs. Two players each take turns in which they take previously unselected vertices in hopes of getting a set of vertices or a structure that gives them the win.

Maker-Breaker games are a type of positional game. It consists of two players who take alternating turns: the *Maker* and the *Breaker*. Maker's winning condition is that they have all the elements of the winning set. Breaker's winning condition is to prevent that goal of Maker, which in other words, is to have at least one element from all of the winning sets. Maker-Breaker games also cannot end in a draw, and in all Maker-Breaker games, one player will have a winning strategy.

In Maker-Breaker games we use $\mathcal{F}$ to denote the group of winning sets for Maker, while we use $\mathcal{F}^*$ to denote the *transversal game* of $\mathcal{F}$, which also happens to be the group of winning sets for Breaker.

Definition 1.1. A *transversal game* is a related game in which the winning sets are the minimal sets that intersect every winning set in the original game.

Each Maker-Breaker game may have a different strategy for either player to guarantee the win. The three different strategies that this paper will explore are the pairing strategy, the strategy from strong positional games, and the potential-based strategy. The pairing strategy is a strategy in which players make disjoint pairs out of the positions in the game. Each turn they take the other position from the same pair that the opponent took in the previous turn if possible. The strong positional strategy involves Maker using the first player's winning strategy in the string positional game variant. The potential-based strategy is where a formula is used to find the value from 0–1 of a winning set. Maker plays to increase the potential sum and Breaker plays to decrease it.

2. A Few Maker-Breaker Games

One of the most well-known Maker-Breaker games is HEX. The board of HEX is a hexagonal grid. The opposite ends of lines on the grid are colored blue, while the other pair of opposite

Date: December 16, 2024.

ends are colored red. Look at Figure 1 for a visual representation. Each player has the goal of coloring a chain of hexagons that connects their respective sides together (blue for Left and red for Right). View Figure 2 to see a completed game in which Blue has won. Each move consists of a player taking up a hexagon that is touching either a hexagon they have already colored or is touching a side of their respective color, and players alternate turns. Something interesting that we can note about HEX is that when the entire board is filled, one and only one of the two players would have won.

Let us first quickly understand how this game is a Maker-Breaker game. We can consider that the winning set consists of all the paths Maker can make from one end to the other. Meanwhile, Breaker's objective is to stop this goal of Maker, and that can be done by having at least one element from the winning sets, which in turn blocks off Maker from getting a win. Its also important to note that the game of HEX cannot end in a draw.

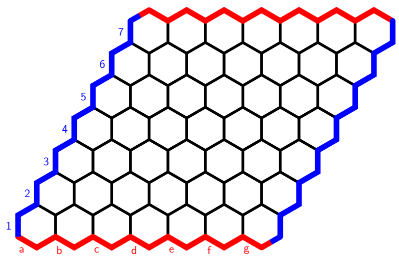

Figure 1. Empty 7x7 HEX grid.

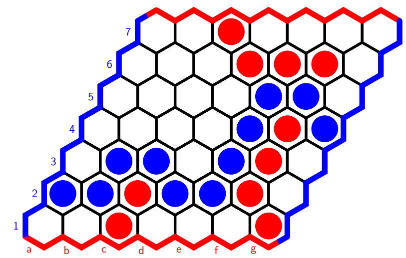

Figure 2. Complete 7x7 HEX grid (Blue win).

Another Maker-Breaker game is SHANNON SWITCHING GAME. Two of the nodes in the game are considered special. Maker's turn consists of coloring any edge that is still on the graph, while Breaker's turn consists of removing any uncolored edge from play. Maker wins if they create a path connecting the two special nodes, while Breaker wins if they prevent this goal of Maker, so to split the graph in two between the two special nodes. View Figure 3 for an example of SHANNON SWITCHING GAME: A and B are the two special nodes; a dotted edge means that the edge is removed, and a blue edge means that the edge is colored. It is easy to see how this game is a Maker-Breaker simply by how the game functions and the rules work. Visit [6] to better understand this particular game and the strategies.

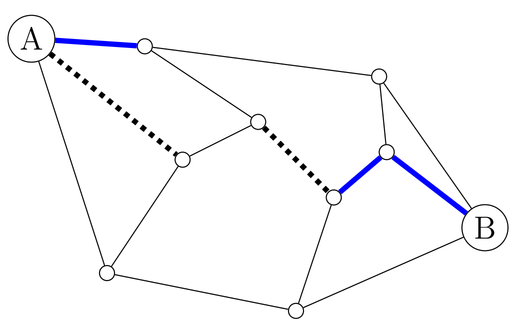

Figure 3. SHANNON SWITCHING GAME diagram

There are many other Maker-Breaker games that we can find, such as TIC-TAC-TOE with the variant that one player, Maker, has the objective of getting 3 tics in a row, while the other player, Breaker, wins if they prevent the goal of Maker. Check [1] to see some other

Maker-Breaker games. However, our purpose is to better understand Maker-Breaker games and to see the implications.

3. Notations

To better understand Maker-Breaker games, we can first consider X to be the set of elements or positions of a Maker-Breaker game. The player who's turn it is to move, makes a move by taking an element of X. Typically, Maker goes first in these games, but Breaker can also play first. Then, the players alternate turns where each player takes an element of X that has not been taken yet. We can also call $\mathcal{F}$ the group of winning sets, which means that it is a group of subsets of X. Recall that Maker wins if they get all the elements of some set in $\mathcal{F}$, and Breaker wins if they get at least one element from each set in $\mathcal{F}$, which would prevent Maker's goal. We use $(X, \mathcal{F})$ to denote Maker-Breaker games.

Something that we can pretty easily see in HEX is that if a player wins going second, they also win when they play first. However, we can find this to be much more generalizable over all Maker-Breaker games.

Theorem 3.1. *Any winning strategy for Maker or Breaker on $(X, \mathcal{F})$ when playing second will also be a winning strategy when playing first.*

Note that this theorem also means that playing first in a Maker-Breaker game is always an advantage.

Proof. This is very easy to prove if we think about how Maker-Breaker games work. From Maker's standpoint, if they win from playing second, which means they took elements of X and eventually formed some set in $\mathcal{F}$, then they can adapt the same strategy when playing first. When originally playing second, Breaker had the benefit of taking the best element of X. When playing first, the benefit is passed to Maker, giving them a further advantage and still guaranteeing the win.

Similarly, if Breaker wins when playing second, they can also adopt the same strategy to win when playing first. Also, they have an even further advantage because the opportunity Maker had of picking the best element in X is now passed on to Breaker, guaranteeing Breaker the win. ∎

Now we will introduce the notation $\mathcal{F}^*$ in which $(X, \mathcal{F}^*)$ is the *transversal game* of $(X, \mathcal{F})$. The winner of $(X, \mathcal{F}^*)$ is always the opposite of the winner in $(X, \mathcal{F})$.

Example. If $\mathcal{F}$ was the group of sets $\{\{A, B, C\}, \{D, E, F\}\}$, then $\mathcal{F}^*$, the transversal game of $\mathcal{F}$, is $\{\{A, D\}, \{A, E\}, \{A, F\}, \{B, D\}, \{B, E\}, \{B, F\}, \{C, D\}, \{C, E\}, \{C, F\}\}$. We can also see that the sets in $\mathcal{F}^*$ consist of two elements, one from each set in $\mathcal{F}$, which is the winning set for Breaker by definition. The same finding can be found for Maker.

4. Strategies

In various Maker-Breaker games, certain strategies can be adopted to guarantee a certain player the win. The first strategy we will be introducing is the **pairing strategy**. In this strategy, players can make disjoint pairs out of the positions on the board. Whenever an opponent takes a position on the board, the player takes the other element in the pair that the opponent's move lies in. For example, consider the following 5 by 5 tic-tac-toe board.

K	A	H	A	L
F	B	B	I	J
C	G		I	C
F	G	D	D	J
L	E	H	E	K

The pairs are the squares marked with the same letter, and we can notice that whenever the opponent makes a move in one of the squares, the player can always take a square that's the pair in order to always end the game in at least a draw. However, in Maker-Breaker games there are no draws, so this functions a little differently.

For Maker, in addition to the rules we have already set for the pairing strategy, all the sets that contain at least one element from each pair must contain some winning set. We can assume that Maker plays second because of Theorem 3.1. Whenever Breaker takes a position, Maker can take the other element in the pair. Thus, by the rules we declared for Maker to form a winning strategy using the pairing strategy, they guarantee themself a win. For Breaker to win using the pairing strategy, there must be the rule that every winning set contains at least one pair. We can also assume that Breaker plays second because of Theorem 3.1, so if Breaker plays by making a move in the other element of each pair that Maker moves in, they would prevent Maker from winning. However, in many Maker-Breaker games, forming these pairs is not possible, and players must use a different strategy in order to win.

The **strategy from strong positional games** is a strategy in which Maker adopts the winning strategy of the first player in the strong positional game variant.

Definition 4.1. A *strong positional game*, which can also be called a *Maker-Maker game*, is a game in which both players try to obtain all the elements of a winning set in the game.

An example of a strong positional game is tic-tac-toe with standard conventions. Note that these games can end in a draw, unlike Maker-Breaker games. We can also see that second cannot win because if they would have won, first could have used the same strategy and won. So we find that these games only have two outcomes: first wins, or second draws.

If we translate these games into the Maker-Breaker version, we can find that there is an advantage for Maker. This is because rather than focusing on obtaining a winning set while also trying to block the other player from getting a winning set, Maker only needs to focus on obtaining the winning set. From there we find that any win for First in a strong positional game is also a winning strategy for Maker in the Maker-Breaker game. The converse is true for Breaker, so every winning strategy for Breaker in a Maker-Breaker game is a drawing strategy for second in the strong positional game variant.

The final Maker-Breaker game strategy we will cover is the **potential-based strategy**. Each winning set is assigned a value for the potential, which is some number between 0 and 1. Whenever Breaker takes an element, the potential of all winning sets with that element becomes 0. Whenever Maker takes a winning set, the potential of that set becomes 1. Also, if Maker takes an element that belongs to a winning winning set that Breaker still has not taken an element from, then the potential of all winning sets with that element increases. If Maker can play so that the potential sum slowly increases, then Maker wins. If Break can play so that the potential sum slowly decreases, then Breaker wins.

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