Sentestrat Scoring in Games With ko rules

Jasmine Li

December 6, 2024

1 Introduction

Further analysis on loopy games can be done by restricting the ability for players to complete a loop. For example, the general ko rule forbids repetition of past board positions. It is adopted from the rulebook in Chinese Go, implemented to prevent preemptive stalemates in scoring. In this paper, we will discover a property that arises in loopy games under a particular ko rule. Then, we will extend scoring to games with ko rules under orthodox-play, so that Hanner and Milnor's Mean Value Theorem will be proven for games with ko rules. Finally, the Mean Value Theorem is used to prove the achievement of a mean value of $\frac{1}{3}$, which is conventionally impossible in the set of short games.

2 An Interesting Property

2.1 **Terminology**

Definition 2.1. Let G be a loopy game. A **ko** is an alternating loop of length 2 in a subposition of G.

Definition 2.2. Definition: We call a loopy game G **simple** if:

- the only loops in G are kos, and
- every subposition H of G has at most one Left option H^L with option $H^{LR} = H$, and at most one Right option H^R with option $H^{RL} = H$.

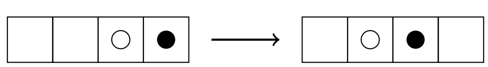
To observe some interesting results in loopy games with ko positions, we introduce a rule on the play of kos. The **ko rule** states that

No position may be repeated during an individual play of G

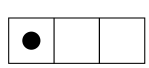
We say a player is **kobanned** from moving to G if the move violates the ko rule.

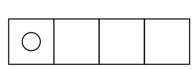
2.2 **A Game Named PUSH**

The game PUSH is played with black and white tokens on a finite strip of squares. Left may push a black token one square to the left, also pushing any tokens in its immediate path. For example, Left may make this move:

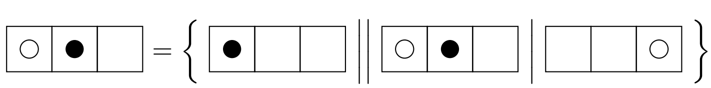


Similarly, Right may push a white token one square to the right, along with any tokens immediately to the right of it. Tokens pushed off the board are removed from the game. Note the scoring for PUSH:

 $= 1,$

 $= -4.$

Consider the following game J :

$J =$  $= \{ 1 \,||\, J \,|\, -1 \}.$

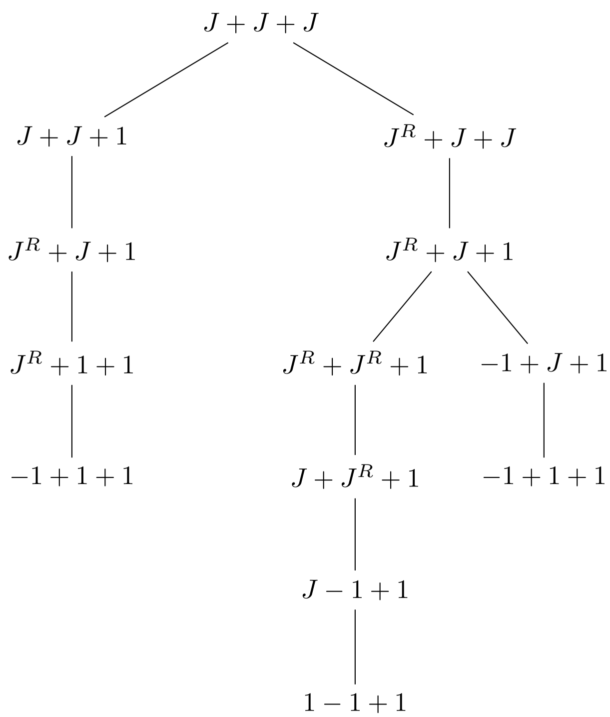
Proposition 2.1. *J is a simple game.*

Proof. We begin by proving all loops in PUSH are kos.
Consider a position P in a PUSH game that is repeated in a loop. There must exist a strip of tokens in P which was pushed left and right an equal amount of times. Because PUSH is a short game, the tokens must be pushed in alternating directions during consecutive moves. Therefore, the loop has length 2.

We finish by proving each player has at most one option in J that is part of a ko.
Without loss of generality, assume that there exists some sub-position P such that Left has two distinct options X, Y which both have P as a Right option. Then, there exists a strip of tokens in X such that Right can push the strip to obtain P. In particular, this is the same strip Left pushed in the move P to X. There must exist a distinct strip of tokens in Y that satisfies the same quality. However, there are at most 1 of each color token in P, so the strips in X and Y cannot be distinct. Ergo, X and Y are the same option. The same argument can be made for options of Right. □

For the rest of the paper, we will analyze gameplay on J and sums with J under the ko rule.

Consider the sum $J' = J + J + J$. We can trace the gameplay of J' to completion as follows:



Remark 2.1. The mean value of J seems to be $\frac{1}{3}$.

We find that $J' = 1$, implying $m(J) = \frac{1}{3}$, which is a value ordinarily unachievable under finite games. Proving this formally is the main result of this paper.

3 Sentestrat for Games with ko rules

Definition 3.1. A **kothreat** is a move to a component with sufficiently large temperature, such that the opponent now favors responding locally to the threat instead of finishing a ko.
Kothreats of size k are denoted as

$$\theta_k = \{\ 2k \mid 0\ \ ||\ \}.$$

The environment in which kothreats are played is called the threat environment.

Definition 3.2. A **threat environment** of size k is denoted

$$\Theta_k = k\theta_k.$$

Definition 3.3. The **Komaster** of a ko is the player who can win the local ko without ignoring a kothreat. Either player prefers to be Komaster, if possible.

The privilege of being Komaster is determined by local auctions before the play begins. If the global game contains several kos, then it is possible for Left to be Komaster of a local ko, and Right to be Komaster of others. See [Ber96] for more.

Consider the sum

$$J + \theta_5$$

when Right moves first to $\{\, J \mid -1 \,\} + \theta_5$.
Left may play on θ_5 to

$$\{\, J \mid -1 \,\} + \{\, 10 \mid 0 \,\}.$$

Right is forced to move in $\{\, 10 \mid 0 \,\}$ to

$$\{\, J \mid -1 \,\}.$$

Left is no longer restrained by the ko rule, so she moves to J, and Right is forced to take a coupon in the environment.

Remark 3.1. Left is the komaster of $J + \theta_5$ because kothreat forces Right to respond locally.

We can now extend sentestrat to games with ko rules.

3.1 Generalized Sentestrat

Generalized sentestrat states:

> Assume that your opponent is komaster.
>
> If your opponent has just moved on a component that is now active at the ambient temperature t, respond locally in that component. Otherwise, play on the component with the hottest temperature below t.
>
> If the suggested move is banned by the ko rule, play on an available component with hottest temperature below t.

Note that the third clause of generalized sentestrat introduces a need for an adjustment in scoring of games with ko rules.

Definition 3.4. Let H be a subposition of a game G played according to sentestrat and let H' be an option of H. Suppose a player makes a move in H to H' at ambient temperature t in accordance to the third clause of generalized sentestrat. Let t' be the temperature of H, such that $t' < t$. The corresponding **ko-adjustment** at H' is given by

$$\Delta u(H') = \begin{cases} t' - t & \text{if } H' \text{ is a left option of } H \\ t - t' & \text{if } H' \text{ is a right option } H \end{cases}$$

Note that in loopfree games, the ko-adjustments are necessarily zero.

Definition 3.5. Let H be a subposition of a game G, and let H' be an option of H. Suppose a player moves from H to H', the **temperature drop** at H', denoted $\Delta t_a(H')$ is given by

$$\Delta t_a(H') = \begin{cases} t_a(H) - t_a(H') & \text{if } H' \text{ is a left option of } H \\ t_a(H') - t_a(H) & \text{if } H' \text{ is a right option of } H \end{cases}$$

4 Orthodox Accounting

In orthodox play, players play for the best score in a game enriched by a coupon stack $\mathscr{E}_t^\delta$ of temperature t and granularity δ:

$$\mathscr{E}_t^\delta = \pm\delta \pm 2\delta \pm 3\delta \pm \cdots \pm (t - \delta) \pm t.$$

In particular, sentestrat recommends only orthodox moves [Sie13]. Orthodox analysis is particularly helpful for Go, wherein the game can be analyzed as a sum of simple components. The components have various values, simulating an enriched environment for a component of interest.

4.1 Stops and Masts

We denote the Left and Right stops of a game G enriched by a coupon stack $\mathscr{E}_t^\delta$ as $L_t^\delta(G)$ and R_t^δ, respectively. Since Left and Right prefer to end at their opponent's stops, the scores of the players are analogous to their respective stops.

Definition 4.1. Let G be a short game with Right as Komaster. A left (resp. right) option G^L (resp. G^R) is said to be orthodox at temperature t if

$$R_t^\flat(G_t^L) - t = L_t^\flat(G_t)$$

(resp. $L_t^\flat(G_t^R) + t = R_t^\flat(G_t)$).

Definition 4.2. The enriched scores $L_t^\delta(G)$ and R_t^δ are given by

$$L_t^\delta(G) = L(G + \mathscr{E}_t^\delta) - \frac{t}{2} \text{ and } R_t^\delta = R(G + \mathscr{E}_t^\delta) + \frac{t}{2}.$$

Definition 4.3. Let G be a simple game.[1] For temperatures $t \geq -1$ we define Left-Komaster scores $L_t^\sharp(G)$ (Left starts) and $R_t^\sharp(G)$ (Right starts), where

$$L_t^\sharp(G) = \lim_{k\to\infty} L_t(G + \Theta_k)$$

$$R_t^\sharp(G) = \lim_{k\to\infty} R_t(G + \Theta_k)$$

and Right-komaster scores $L_t^\flat(G)$ and $R_t^\flat(G)$ as follows:

$$L_t^\flat(G) = \lim_{k\to\infty} L_t(G - \Theta_k)$$

$$R_t^\flat(G) = \lim_{k\to\infty} R_t(G - \Theta_k).$$

Definition 4.4. We call $m^\sharp(G)$ the mast value for games G in which Left is Komaster. $m^\flat(G)$ is the mast value of games G with Right as Komaster.

Proposition 4.1. $m^\sharp(G) \geq m(G) \geq m^\flat(G)$

Proof. Consider the game $G + \Theta_k$. Note that Right is given no new options on G. Thus,

$$L_t^\delta(G + \Theta_k) \geq L_t^\delta$$

for all δ and t. Then, for when $L_t^\sharp(G)$ is defined,

$$L_t^\sharp(G) \geq L_t(G).$$

Similarly, the game $G - \Theta_k$ is at most as favorable to Left as G, so $L_t(G) \geq L_t^\flat(G)$. The same argument works for Right. Consequently, we have

$$L_t^\sharp(G) \geq L_t(G) \geq L_t^\flat(G),$$
$$R_t^\sharp(G) \geq R_t(G) \geq R_t^\flat(G).$$

The proposition then follows. □

If a loopy game G contains a ko that is hotter than G, then G is particularly sensitive to kothreats. Necessarily in these games, $m^\sharp(G) \neq m^\flat(G)$.

Definition 4.5. A game G **hyperactive** if a subposition of G contains a ko hotter than G itself, **placid** otherwise.

Consequently, for placid games G, we have $m^\sharp(G) = m^\flat(G)$.

Proposition 4.2. *J is a placid game.*

Proof. The only subposition of J that is a ko is J itself. Then, J is not hyperactive. □

[1]For simple games, these limits exist. See [Sie13] for more.

4.2 Extension of the Orthodox Accounting Theorem

Definition 4.6. Let t be the ambient temperature of the sum $G_1+\cdots+G_k$. The **Left and Right orthodox forecasts** at temperature t are given as

$$x_t = m(G_1) + \cdots + m(G_k) + \frac{t}{2}$$
$$y_t = m(G_1) + \cdots + m(G_k) - \frac{t}{2},$$

respectively.

Theorem 4.1. *(Orthodox Accounting Theorem) Let $G_1, ..., G_k$ be simple and placid. Suppose Left plays first (resp. second) on the sum*

$$G = G_1 + \cdots + G_k,$$

following sentestrat*. She is guaranteed a score of at least*

$$z + \frac{1}{2}\sum_{i=1}^{w} \Delta t_i + \sum_{j=1}^{n} \Delta u_j,$$

where z is the Left (resp. Right) orthodox forecast, Δt_i are the temperature drops, and Δu_j are the ko-adjustments.

Proof. We proceed by induction on the number of moves made on G for the score of Left.
We let the ambient temperature t at the start be defined as follows:

$$t = \max(0, t^\flat(G_1), ..., t^\flat(G_k).$$

This is sufficient because Left assumes Right is komaster, and the first component Left moves on must have temperature $\leq t$, ideally as close to t as possible.

Let n' denote the number of ko-adjustments made on G so far in a play. We will show that for each subposition $Y = Y_1 + \cdots + Y_k$ of G with Right to move

$$R_t^\flat(Y_1) + R_t^\flat(Y_2) + \cdots + R_t^\flat(Y_k) - \frac{t}{2} + \sum_{j=1}^{n'} \Delta u_j \geq y_t \tag{1}$$

is satisifed, provided that at least one component Y_i is active at t. Furthermore, each subposition $X = X_1 + \cdots + X_k$ with Left to move satisfies

$$L_t^\flat(X_1 \backslash Z) + R_t^\flat(X_2) + \cdots + R_t^\flat(X_k) + \frac{t}{2} + \sum_{j=1}^{n'} \Delta u_j \geq x_t, \tag{2}$$

such that at least one component X_i is active at t, where Right just moved from Z to X_1.

Consider the time when no moves have been made on G. The number of ko-adjustments n' on G is 0. Evidentally, we have $R_t^\flat(G_1)+R_t^\flat(G_2)+\cdots+R_t^\flat(G_k)-\frac{t}{2} \geq y_t$ and $L_t^\flat(G_1)+R_t^\flat(G_2)+\cdots+R_t^\flat(G_k)+\frac{t}{2} \geq x_t$, since $R_t^\flat(G_i) = m(G_i) = L_t^\flat(G_i)$. This will be our base case.
Assume (1) and (2) are true for positions where the number of moves N is less than a certain p.

Consider a subposition Y satisfying (1). Such Y exists due to the inductive hypothesis. Without loss of generality, assume Right moves on the component Y_1 to Y_1^R. Let the resulting position be $X = Y_1^R \backslash Z + Y_2 + \cdots + Y_k$, where $Z = Y_1$. Therefore,

$$L_t^\flat(Y_1^R \backslash Y_1) + t = R_t^\flat(Y_1)$$

by **Definition 4.1**. Then (2) follows.

Consider a subposition $X = X_1 + X_2 + \cdots + X_k$ which satisfies (2). There are two cases for X_1's activity at t.

Case 1: X_1 is active at t. There are two subcases.

- **Case 1a:** If there is some $X_1^L \neq Z$ which sentestrat recommends, then by **Definition 4.1** we have
$$R_t^\flat(X_1^L) - t = L_t^\flat(X_1),$$
from which the recurrence (1) follows.

- **Case 1b:** Otherwise, sentestrat recommends a move elsewhere, say from X_2 to X_2^L. Because this move follows from the third clause of generalized sentestrat, there is a ko-adjustment:
$$\Delta u(X_2^L) = L_t^\flat(X_2) - R_t^\flat(X_2^L) + t \geq R_t^\flat(X_2) - R_t^\flat(X_2^L) + t.$$
Furthermore, if Z is kobanned, then
$$L_t^\flat(X_1 \backslash Z) = R_t^\flat(X_1),$$
since neither player can move to Z. Therefore,
$$R_t^\flat(X_1) + R_t^\flat(X_2^L) + \Delta u(X_2^L) \geq L_t^\flat(X_1 \backslash Z) + R_t^\flat(X_2) + t,$$
from which (1) follows where $Y_2 = X_2^L$.

Case 2: X_1 is dormant at t. Then, sentestrat recommends a move elsewhere, say on active component X_2 to X_2^L. Therefore,
$$R_t^\flat(X_2^L) = L_t^\flat(X_2) + t \geq R_t^\flat(X_2) + t,$$
from which we find
$$\begin{aligned}
&R_t^\flat(X_1) + R_t^\flat(X_2^L) + \cdots + R_t^\flat(X_k) - \frac{t}{2} + \sum_{j=1}^{n'} \Delta u_j \\
&\geq R_t^\flat(X_1) + R_t^\flat(X_2) + t + \cdots + R_t^\flat(X_k) - \frac{t}{2} + \sum_{j=1}^{n'} \Delta u_j \\
&\geq y_t,
\end{aligned}$$
which satisfies (1). If the recommended move is kobanned, then the situation is equivalent to Case 1b.

Eventually all the components will become dormant at the ambient temperature t, say at position Y. Players will take coupons from the environment until the temperature of a coupon dips below that of some component Y_i. (Without loss of generality, let this component be Y_1 with temperature t', which becomes the new ambient temperature.) If this happens after Left moves, then the temperature drop is calculated as
$$\Delta t(Y_1) = t - t'$$
Let X be the position reached after the temperature drop, where $X_1 = Y_1^L$, and $X_i = Y_i$ for $i > 1$. We have
$$m_{t'}^\flat(X_1) + m_{t'}^\flat(X_2) + \cdots + m_{t'}^\flat(X_k) - \frac{t}{2} \geq y_t.$$
By induction on G, since Left goes second on X, she is guaranteed a score of at least
$$\begin{aligned}
&m_{t'}^\flat(X_1) + m_{t'}^\flat(X_2) + \cdots + m_{t'}^\flat(X_k) - \frac{t'}{2} + \frac{1}{2}\sum_{i=2}^{w} \Delta t_i + \sum_{j=1}^{n'} \Delta u_j \\
&= m_{t'}^\flat(X_1) + m_{t'}^\flat(X_2) + \cdots + m_{t'}^\flat(X_k) - \frac{t}{2} + \frac{1}{2}\Delta t(Y_1) + \frac{1}{2}\sum_{i=2}^{w} \Delta t_i + \sum_{j=1}^{n'} \Delta u_j \\
&\geq y_t + \frac{1}{2}\sum_{i=1}^{w} \Delta t_i + \sum_{j=1}^{n'} \Delta u_j
\end{aligned}$$
where t_i represents the ith temperature drop, and w' the total number of temperature drops through the play. The same argument can be made if the temperature drop happens after a move done by Right, yielding X where Left goes first. □

4.3 Mean Values

We now present two important corollaries of the *Orthodox Accounting Theorem.*

Theorem 4.2. *Let $G_1, ..., G_k$ be simple and placid games. Suppose Left plays first (resp. second) on the sum*

$$G = G_1 + ... + G_k + \mathscr{E}_t^\delta$$

and follows sentestrat*. Then she is guaranteed at least the Left (resp. Right) orthodox forecast, to within a bounded multiple of δ.*

Proof. Let $s(G)$ be the number of subpositions in G, and let $g = G_1 + \cdots + G_k$. For each temperature drop Δt_i, the number of moves on g since the preceding temperature drop is bounded by $s(G)$. Moreover, at most $s(G) + 1$ moves were played on the coupon stack $\mathscr{E}_t^\delta$ since the last drop. So, we have

$$\sum_{s(G)\geq 1} |\Delta t_i| \leq 2 \cdot s(G) \cdot \delta,$$

Successive temperature drops happen when $s(G) = 0$, and alternate between positive and negative for Left and Right. There can be at most $s(G)$ moves on g, so

$$\sum_{s(G)=0} \Delta t_i \geq -s(G) \cdot \delta.$$

Then, we obtain the bound

$$\sum \Delta t_i \geq -3 \cdot s(G) \cdot \delta.$$

Similarly, the number of ko-adjustments is bounded by $s(G)$. Each ko-adjusment has magnitude of at most $|s(G)\delta|$. Therefore,

$$\sum \Delta u_j \geq k \cdot s(G)\delta$$

for some constant $s(G) \geq k \geq -s(G)$. From the Orthodox Accounting Theorem, it follows that Left going first (resp. second) is guaranteed the Left (resp. Right)orthodox forecast within a bounded multiple of δ. □

Theorem 4.3. *(Mean Value Theorem) Let $G = G_1 + \cdots + G_k$ is a sum of simple placid games. Then,*

$$m(G) = m(G_1) + \cdots + m(G_k).$$

Proof. Let $x = m(G_1) + \cdots + m(G_k)$. The *Orthodox Accounting Theorem* and Theorem 4.2. show that for $t = \max(0, G_1, G_2, ..., G_k)$, the enriched scores $L_t^\delta(G)$ and $R_t^\delta(G)$ satisfy the inequalities

$$|L_t^\delta(G) - x| \leq k\delta \text{ and } |R_t^\delta(G) - x| \leq k\delta,$$

for some constant k. By the definition of enriched scores and application of the definition of a limit, $L_t(G) = R_t(G) = x = m(G)$ for all such t. □

We return to main result of the paper.

Proposition 4.3. $m(J) = \frac{1}{3}$.

Proof. By the Mean Value Theorem,

$$m(J') = m(J) + m(J) + m(J)$$
$$\implies m(J) = \frac{1}{3}.$$

□

5 Conclusions

This paper extends two fundamental theorems of combinatorial game analysis on short games to loopy games under ko rules (allowing finite plays). The classical game Go, has such a set of ko rules. Go had been of particular interest due to its resistance against artificial intelligence domination (in 2016, a human grandmaster defeated a computer in a match). Generalization of scoring to include games with the ko rule has been used for evaluation of end-stage positions of Go, at which point interactions between various parts of the board are limited, allowing analysis of a board as a sum of simple components. Mathematical analysis of Go has many applications in economics, politics, psychology, and evolutionary biology.

References

[Ber96] Elwyn Berlekamp. "The Economist's View of Combinatorial". In: *Games of No Chance, SLMath Publications* 29 (1996), pp. 365–405.

[Sie13] Aaron Siegel. "Combinatorial Game Theory". In: *Graduate Studies in Mathematics* 146 (2013), pp. 333–396.