MANCALA-LIKE GAMES

Jackson Brown

Abstract. Certain endgame positions in the combinatorial Mancala-type game Ayo, can be related to winnable positions in the solitaire Mancala-type game Tchoukaillon, which is solved. We will show how to recognize and win these determined endagme positions. The work is mostly sourced from Duane Broline and Daniel Loeb's paper *The combinatorics of Mancala-type games* [BL95], although it has been altered to concur with standard Combinatorial Game Theory nomenclature.

1 Inroduction

Mancala is a family of combinatorial games which can further be classified as a family of sowing games and usually scoring games. In this paper we review some basic results regarding a Nigerian variant, Ayo, and a European solitaire variant, Tchoukaillon. We will show how an understanding of winnable positions in Tchoukaillon allows one to optimally play certain Ayo endgames, called *determined positions*, and provide an algorithm for generating all such winning positions and determined positions.

2 Tchouka and Tchoukaillon

The game Tchouka is a Russian combinatorial game played with a number of ordered pits, each initially containing a certain number of stones, and an additional empty pit called the Rouma, Cala or Roumba. (See Figure 1. Note that the board may be arranged in a circular arc. Still, there must be a defined direction and ordering of the pits that terminates at the Roumba.)

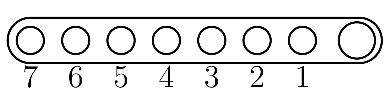

Figure 1. A Tchouka or Tchoukaillon board

One chooses a pit to sow the stones from, taking all the stones and distributing them one at a time in the succeeding pits in the direction of the Roumba. If necessary, the sowing continues, wrapping around to the pit furthest from the Roumba.

When played as a two-player game, each player continues until he is forced to place a last stone in an empty pit. At that point, his opponent moves. The game is a scoring game; each player attempts to move more stones in the Roumba than the other.

In the solitaire variant, called Tchoukaillon, the last stone *must* land in the Roumba, and no wrap-around moves are allowed.

The goal is to move all the stones into the Roumba. Not all positions are winning.

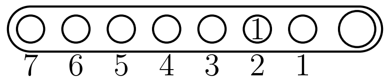

Figure 2. An unwinnable Tchoukaillon position; there is one stone in the second pit from the Roumba.

Consider Figure 2. There is only one option and it leaves a stone on the board with no further options available. A position from which a win is possible is called a *winning position* Because harvesting pit 1 in Figure 2 would not permit further play, we say it is not harvestable.

Some pits are *harvestable*, and some are not.

Definition 2.1. *In* Tchoukaillon, *pit i is* harvestable *if and only if it contains i stones, so that when harvested the last stone lands in the Roumba.*

Definition 2.2. *In* Tchoukaillon,*a a pit i containing j stones is* underfull *if $j < i$, and* overfull *if $j > i$.*

Pits that are underfull may become harvestable after some play, but pits which are overfull will never become harvestable, as a pit can only lose stones by being harvested.

Lemma 2.1. *A winning position in* Tchoukaillon *cannot contain overfull pits.*

Proof. If a Tchoukaillon position is winning, it must reach a state where all of its pits are empty, by definition. If a pit is overfull, it cannot be harvested because it has too many stones. A pit can only lose stones by being harvested. Then this overfull pit can never be harvested and the board can never be cleared, so the position cannot be won. ■

3 Ayoyayo

The game Ayoyayo, or simply Ayo, is played on a board of two rows, each consisting of six pits which initially contain four stones each. *L*eft, who owns the bottom, "*L*ow" row, and *R*ight, who owns the top, "*R*aised" row, alternate turns, and the objective is to capture as many stones as possible; it is a combinatorial scoring game. To move, a player chooses a non-empty pit in their row and sows it just as in Tchouka.

If the last pit sown by a player is in the opponent's row, and contains (after having been sown) two or three stones, then the stones in the pit are captured by the active player. The stones from consecutively preceding pits which meet the same criteria are also captured.

There is also a special condition of a pit, called an *Odu*[1], which will not come up in any positions we consider.

At each turn, a player must, if possible, move in such a way that their opponent will be left with a legal move. If, on a turn, a player cannot move in such a way, the game is over and the player is awarded all remaining stones on the board.

If there are so few stones left on the board that neither player can ever capture, but both players will always have a move–such that the game, ignoring scoring, would become

[1] This occurs when, on a board which has n pits on each side, a given pit has n stones or more. [Ode77]

an endless cycle with no captures–the game is over and each player is awarded the stones in their own row.

We shall analyze a specific type of endgame in a generalized Ayo game where the board contains n pits in each row. We will, somewhat unintuitively but for reasons that will later become apparent, number the pits from $-n+2$ to $n+1$, moving clockwise. Left will own the pits numbered 2 through $n+1$ and Right will own the pits numbered $-n+2$ through 1. Sowing moves in the direction from higher numbered pits to lowered ones, and from pit $-n+2$ to pit $n+1$

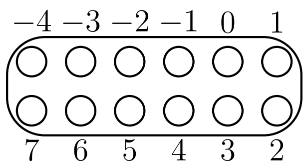

Figure 3. An Ayo board with two rows of 6 pits each, in standard labeling.

We will study the endgame positions known as *determined positions.*

Definition 3.1. *A* determined position *is a position on a* $2n$*-pit* Ayo *board where it is possible for a player, WLOG Left, to move such that:*

- *Left captures at every turn.*
- *After every turn, Right has only one stone in his row and thus one option.*
- *Every stone is captured by Left except for one which is captured by Right.*

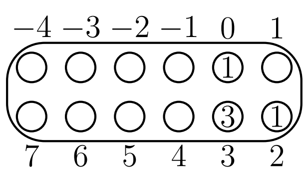

Figure 4. A determined position in Ayo with Left to move, capturing from pit 3

Lemma 3.1. *The stone in Right's row in a determined* Ayo *position must be in pit 1 if Right is to move and in pit 0 if Left is to move.*

Proof. If Left is to move, she must capture and leave only one stone in Right's row. Thus the captured stone must lie in Right's second pit (pit 0). Hence before Right's move, the stone must have been in pit 1. ∎

4 Applying Tchoukaillon to Ayo

There is a close relationship between Tchoukaillon and Ayo. Suppose Left and Right play Ayo on a board of size $2n$. If we were to focus on pits $1, 2, ..., n+1$, we would have the impression we were watching a game of Tchoukaillon with pits numbered $1, 2, ..., n+1$.

Theorem 4.1. *There is a one-to-one correspondence between determined positions in* Ayo *and winning positions in* Tchoukaillon*. To find the* Tchoukaillon *position corresponding to a determined* Ayo *position, ignore pits* $0, -1, -2, ..., -n+2$ *and focus on pits* $1, 2, ..., n+1$.

Proof. Let D be a determined Ayo position corresponding to the Tchoukaillon position W.

- If pit 1 has a stone, then by Lemma 3.1, Right must move the stone from pit 1 to pit 0. The corresponding move in Tchoukaillon is legal: remove a stone from pit 1.
- If pit 1 is empty, then by Lemma 3.1, Left must capture the stone in pit 0. She does so by harvesting some pit i containing exactly i stones, placing stones in pits $1, 2, ...i-1$, and capturing the stone in pit 0. The corresponding move Tchoukaillon is legal: harvest some pit i containing i stones, placing stones in pits $1, 2, 3, ..., i-1$, and pocketing the remaining stone in the Roumba.

Similarly, the legal moves in W correspond to Left's options which are determined moves in D. The objective of Tchoukaillon is to empty the board. At the end of determined Ayo play, the board is empty except for pit 0–which is not a part of our focus. A legal Tchoukaillon move ends in the Roumba, and a determined Ayo move for Left ends in the corresponding pit 0.

Thus, W is winning if and only if D is determined. ■

Now that we have shown equivalence, we may analyze Tchoukaillon with the understanding that the results will describe optimal play in a determined Ayo position.

5 Winning Tchoukaillon

Theorem 5.1. *If a win is possible from a given* Tchoukaillon *position, the unique winning move is to harvest the smallest harvestable pit.*

Proof. Suppose pits i and j are harvestable and, WLOG, $i < j$. If pit j is harvested, then pit i will contain $i+1$ stones, and be overfull. By Lemma 2.1 the position could then never be won. ■

This reasoning can be applied in reverse; that is, given a winning Tchoukaillon position, A, with s stones, we can obtain a winning position, B, with $s+1$ stones, as follows:

- Let $i \geq 1$ be the the least number such that pit i is empty.
- Place i stones in pit i.
- Remove one stone from each pit j where $j < i$.[2]

Using this procedure, we can show the following:

[2] By definition of i, these pits are all non-empty.

Theorem 5.2. *For all $s \geq 0$, there is exactly one winning* Tchoukaillon *position with s total stones.*

Proof. We have added i stones and removed $i-1$ stones, so B has 1 more stone than A. The winning strategy is to harvest the smallest harvestable pit. None of the pits in front of pit i can be harvestable–if some pit j, where $j < i$, is harvestable in B and so has j stones, that same pit must have had $j+1$ stones in A, but this would mean it were overfull, but by Lemma 2.1, A, which is a winning position, cannot have overfull pits. Then, the unique winning move in B is to harvest pit i, which brings us back to A. We have then obtained an explicit bijection between winning positions with s stones and those with $s+1$ stones. Because there is only one winning position with 0 stones, by induction there is exactly one winning position with s stones for all non-negative integers s. ■

Here are the first few.

Stones s	Pit 7	Pit 6	Pit 5	Pit 4	Pit 3	Pit 2	Pit 1
0							
1							1
2						2	
3						2	1
4					3	1	
5					3	1	1
6				4	2		
7				4	2		1
8				4	2	2	
9				4	2	2	1
10			5	3	1	1	
11			5	3	1	1	1
12		6	4	2			
13		6	4	2			1
14		6	4	2		2	
15		6	4	2		2	1
16		6	4	2	3	1	
17		6	4	2	3	1	1
18	7	5	3	1	2		
19	7	5	3	1	2		1
20	7	5	3	1	2	2	
21	7	5	3	1	2	2	1

Table 1. The first 21 winning Tchoukaillon positions, all that can fit on a 7-pit board.

6 Further Reading

The periodicity of whether of the number of stones a given pit has in the winnable position with s stones as s increases is rather interesting, and the maximum number of stones that a determined position with a given number of pits can hold is bounded by an expression that quite surprisingly includes pi. More can be found in [BL95].

7 Remarks

Many thanks to Euler Circle and Simon Rubenstein-Salzedo for overseeing the Combinatorial Game Theory class through which this paper was produced, as well as to David Walton for being a TA for the class and providing advisement and feedback.

References

[BL95] Duane M. Broline and Daniel E. Loeb. *The combinatorics of Mancala-type games: Ayo, Tchoukaitlon, and 1/pi*. 1995. arXiv: `math/9502225 [math.CO]`. URL: `https://arxiv.org/abs/math/9502225`.

[Ode77] A. O. Odeleye. *Ayo: A popular Yoruba game*. Oxford University Press, Ibadan, Nigeria (1977).