ON THE GREEN JUNGLE AND THE PURPLE MOUNTAIN

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ABSTRACT. Welcome to the *Green Jungle*! Our journey awaits to explore a variant of a famous combinatorial game: GREEN HACKENBUSH. Unlike the case of traditional HACKENBUSH, the first player can always find a winning strategy in GREEN HACKENBUSH if the initial position allows for it (e.g the starting configuration is not a $\mathcal{P}-$position). We can do this by inviting another well-known combinatorial game: NIM. Prepare to survive the jungle as we encounter bamboo stalks, towering trees, and mysterious creatures as the night unfolds!

1. INTRODUCTION

Combinatorial games are a set of games that have a set of (loosely) defined characteristics. Whenever we change a small part of the game (either how the players can move on their given turn or how the game is won by a player) it creates an entirely new game to study. For our purposes, we will be only dealing with games that are played with *exactly* two people. We name these players Left and Right, where the pronoun of Left is she and the pronoun of Right is he for convenience.

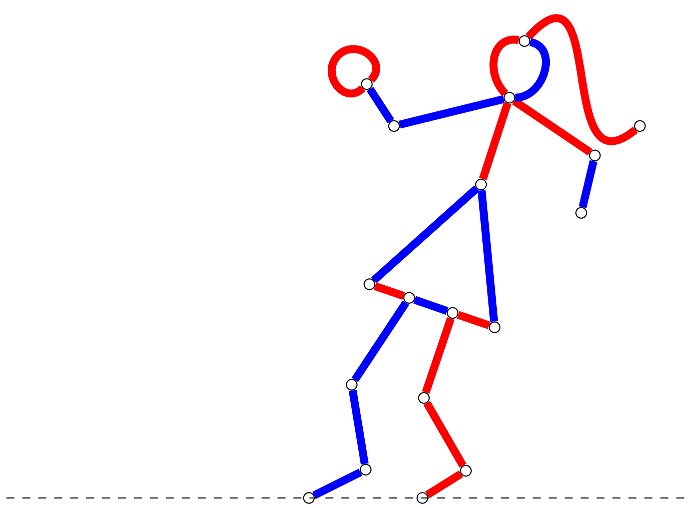

Figure 1. A sample HACKENBUSH game known as HACKENGIRL.

Today, we will be focusing on a variant of HACKENBUSH, traveling to the great green jungle! Before we begin, note that for any game, we have there to be four different *outcome classes*:

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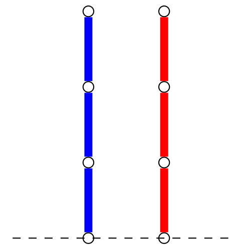

Figure 2. Red-Blue Bean Stalks

Definition 1.1 (Outcome Classes)**.** Denote a game G. The four possible *outcome classes* of the game are $\mathcal{L}, \mathcal{R}, \mathcal{N}$, and $\mathcal{P}$:

- If $G \in \mathcal{L}$, then Left wins regardless of which player goes first.
- If $G \in \mathcal{R}$, then Right wins regardless of which player goes first.
- If $G \in \mathcal{N}$, then the player who starts will win the game.
- If $G \in \mathcal{P}$, then the player who starts will lose the game.

Outcome classes precisely tell us the information that is needed to understand who will win a game. Before we analyze the variant of HACKENBUSH, we first need to be familiar on how to play (regular) HACKENBUSH:

Definition 1.2 (Hackenbush)**.** The combinatorial game HACKENBUSH is played with two players named Left and Right. At the start of the game, a graph with blue and red edges are connected to a specified ground (we will be denoting the ground as a dashed line). The two players take turns making moves. When it is Left's turn to make a move, she will remove any remaining blue edge. When it is Right's turn to move, he will remove any red edge. Moreover, if a move by either player results in edges of the graph not being connected to the ground via other edges, the edge is automatically deleted. A player who cannot make a move on their turn loses.

From now on, we refer to (regular) HACKENBUSH as just HACKENBUSH.

Consider the HACKENGIRL board shown in Figure 1. Notice that the initial game is a graph such that each edge is connected to the ground via other edges and each edge is either blue or red. As this is a game, a natural question is *Who will win?*. Actually, Figure 1 is a quite complicated HACKENBUSH board to analyze. Instead, let us look at some colorful "bean stalks" to get a better understanding of HACKENBUSH.

Consider the HACKENBUSH board shown in Figure 2. Who will win now? Notice that if Left goes first, Right can simply mirror the move that Left does. Hence, Right will always win when Left plays first. Similarly, Left will always win if Right plays first. Hence, we know that when Left plays first, Right will win, and vice-versa. By Definition 1.2, we have that the outcome class for this game is $\mathcal{P}$. The type of strategy being implemented by the players to win is referred to as a **mimicking strategy**, as whatever a particular player does, the other player will simply do the same move with respect to their possible moves. This strategy is a very common way to study combinatorial games, and we will see this strategy appear again in our paper. Finally, we refer to games with outcome class as $\mathcal{P}$ as **zero games**. The reason for the name is due to theory of assigning numbers to games. As this topic is a diversion to our focus, we omit further details for the reason of the name.

Even though HACKENBUSH played with red and blue edges has a lot of depth to their analysis, in this paper we wish to completely alter the view of the game board and limit the game board to having only one color. This color is green and welcome to the world of GREEN HACKENBUSH! In the next section, we will start our exploration into the green jungle by analyzing this game, and by the end of the section, we will be able to completely master (i.e. win) any game board of this game given that the intial position of the game is not a $\mathcal{P}-$position.

2. Exploring the Green Jungle

Our exploration is based on the results of [Fer02].

Up till now, we have talked about HACKENBUSH when there are two colors (namely, red and blue). Now, for the star of our paper, we will be introducing GREEN HACKENBUSH.

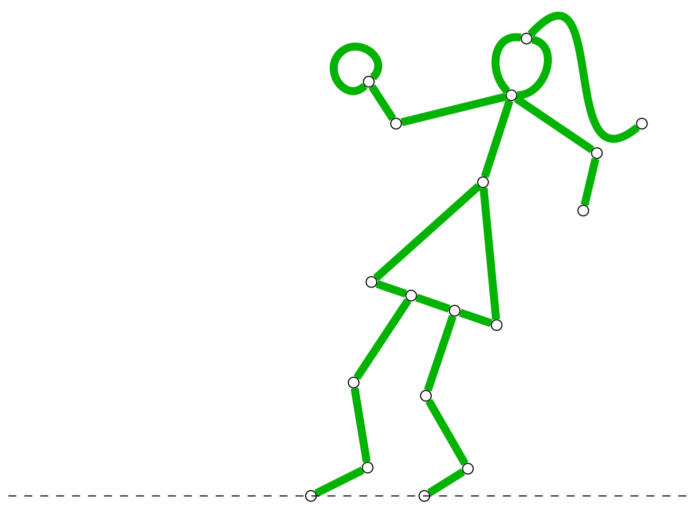

Figure 3. GREEN HACKENGIRL.

Definition 2.1. (Green Hackenbush) The combinatorial game GREEN HACKENBUSH is played with two players named Left and Right. At the start of the game, a graph with green edges are connected to a specified ground (we will be denoting the ground as a dashed line). The two players take turns making moves, and both players are allowed to pick any of the edges of the graph on their turn. Moreover, if a move from either of the players results in a edges of the graph not being connected to the ground via other edges, the edge is automatically deleted. A player who cannot make a move on their turn loses.

For example, Figure 3 is the equivalent game board in GREEN HACKENBUSH to Figure 1 in HACKENBUSH.

Before we go any further in our discussion, we would like to note a rather important difference between HACKENBUSH and GREEN HACKENBUSH, in particular, comparing the types of moves a player has:

Definition 2.2 (Partizan Games)**.** A *partizan game* is a combinatorial game in which each player has a different set of moves from a particular position.

Definition 2.3 (Impartial Games)**.** An *impartial game* is a combinatorial game in which each player has the same set of moves available from any given position.

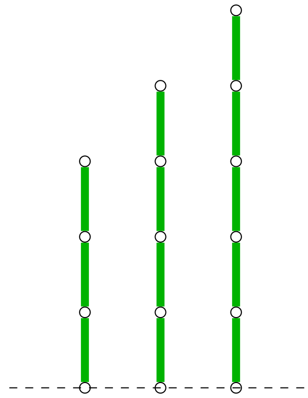

Figure 4. Bamboo Stalks

From the above two definitions, we see that HACKENBUSH is an example of a partizan game, while GREEN HACKENBUSH is an example of an impartial game. Also, even though partizan games can have any of the four outcome classes, impartial games can only be an $\mathcal{N}$ position or $\mathcal{P}$ position. This is due to the fact that since Left and Right have the same set of moves, there should be no reason that Left (or Right) always wins. Hence, when analyzing GREEN HACKENBUSH, we have to only consider $\mathcal{N}$ and $\mathcal{P}$ outcome classes.

2.1. **Bamboo Stalks.** Now, we are finally ready to start our exploration into the green jungle (GREEN HACKENBUSH), with our intention to survive the jungle (win the game). We will do this by a series of baby steps to work up to a generalization. At the moments we enter the jungle, we see *bamboo stalks*:

Definition 2.4. A *bamboo stalk* with n segments is a vertical line graph of n edges with the bottom edge rooted to the ground.

For example, if our "jungle" consists of three bamboo stalks of size $3, 4$ and 5, we will have the configuration shown in Figure 4.

Encountering rocks and stones throughout the jungle, the board shown in Figure 4 reminds us of another famous combinatorial game; to be exact, the impartial game NIM.

Definition 2.5 (Nim)**.** The combinatorial game NIM is a two player game in which the players alternate removing at least one stone from a single pile of stones. The players start with several piles of stones and the player who removes the last stone is the winner.

Notice that NIM is also an impartial game because each player has the same set of moves at a given position, they are free to pick any number of stones from any pile.

Now, we notice that each bamboo stalk of n segments is equivalent to a nim pile with n stones. Let's establish this bijection for a generalized game of GREEN HACKENBUSH with bamboo stalks and NIM: if there are k piles of stones and each pile contains a_i stones for $1 \leq i \leq k$, then in GREEN HACKENBUSH there would be k bamboo stalks and the i^{th} bamboo stalk will contain a_i green edges. In a similar fashion, we can go from a GREEN HACKENBUSH position to a NIM position. Now, for the actual play, if a certain player takes s stones from pile p, then this is equivalent to choosing the s^{th} bamboo stalk edge from the top of the p^{th} bamboo stalk, since then all the edges above them will also disappear (they are not connected to the ground anymore). Moreover, if a player cuts the e^{th} edge from the top of the b^{th} bamboo stalk, then this will result in that bamboo stalk losing e edges, which is equivalent to the b^{th} NIM pile losing e stones. Hence, we have that when analyzing a GREEN HACKENBUSH game with all bamboo stalks, we can treat this as a game of NIM.

This equivalence is great because we know how to win NIM! Let us return to Figure 4, representing NIM with three piles of $3, 4$ and 5 stones. Now, to determine the outcome class of the game, we must take evaluate their nim sum: $3 \oplus 4 \oplus 5 = 2$. *But what can we do with this?* We have that Theorem 2.6 tells us how the value from the nim-sum is vital.

Theorem 2.6. *The outcome class of a game of* NIM *with piles of size* $a_1, \cdots, a_k$ *is a* $\mathcal{P}$ *position if and only if*

$$a_1 \oplus a_2 \oplus \cdots \oplus a_k = 0.$$

This result was first established by Charles L. Bouton in 1901 [Bou02]. We are omitting the proof of 2.6 as it needs ideas outside the scope of this paper, but an interested reader can check Bouton's paper.

As mentioned before, we have that impartial games can only have two outcome classes: $\mathcal{N}$ and $\mathcal{P}$. Hence, we have the outcome class of Figure 4 is $\mathcal{N}$ (we got a non-zero nim sum). Moreover, beyond just finding the outcome classes of GREEN HACKENBUSH with all bamboo stalks, thanks to the equivalence to NIM, we can see how to win this game too. Going back to our example, we have that their nim sum is:

$$\begin{array}{cccccc} 3 & & 0 & 1 & 1 \\ 4 & & 1 & 0 & 0 \\ 5 & \oplus & 1 & 0 & 1 \\ \hline 2 & & 0 & 1 & 0 \end{array}$$

Now, to find the winning moves, we have that select the leftmost place in the nim sum that has a 1. In our case, we have that there is only one place with a 1. In that place, we have only one choice (since there is only one pile with a 1 at that place). Hence, the only winning move is replacing the pile of 3 stones with a pile of $3 \oplus 2 = 1$ stones. Note that $1 \oplus 4 \oplus 5 = 0$, and hence this game now a $\mathcal{P}-$position. Since this is a $\mathcal{P}$ position upon the start of the second player, the first player will win. Hence, if our GREEN HACKENBUSH game starts will all bamboo stalks, we can win the game.

2.2. **Green Trees.** Till now, we were talking about bamboo stalks. Next up in our jungle, we see some green tress! This is our second class of GREEN HACKENBUSH we will analyze. We will first need to define what a *rooted tree*:

Definition 2.7 (Rooted Trees)**.** A *rooted tree* is a graph such that there exists a distinct vertex (which we call the root) and there is a unique path from every vertex to the root.

Definition 2.8 (Green Hackenbush on Trees)**.** A GREEN HACKENBUSH game which contains different disjoint rooted tree graphs and the roots of the trees are the only vertices on the ground.

For example, Figure 5 is an example of a starting position of GREEN HACKENBUSH on trees with a single tree.

As done in the previously, we have already been able win GREEN HACKENBUSH on bamboo stalks, but we would like to extend this to GREEN HACKENBUSH on trees. We can do this with the help of the *Colon Principle*, but before we do that, we would like to define a notation which inspires the name of the principle. For arbitrary graphs G and H with G having an

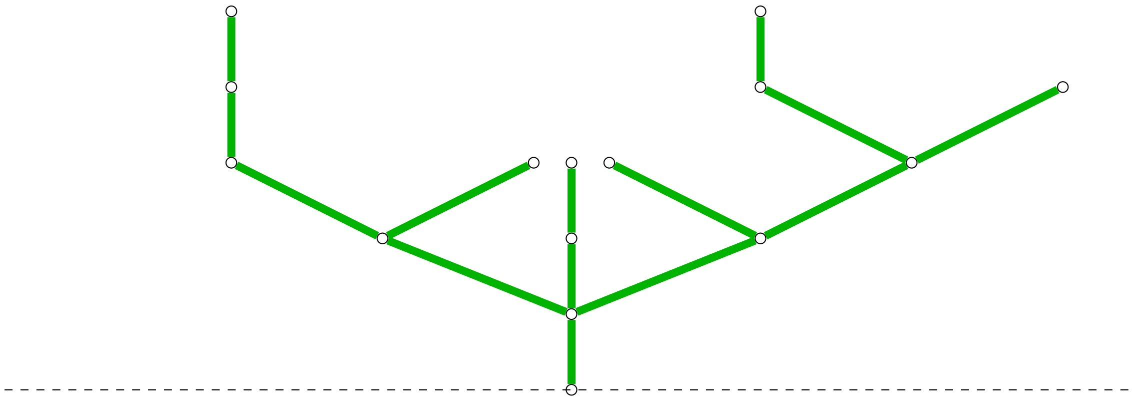

Figure 5. A GREEN HACKENBUSH tree

arbitrary vertex v, we denote $G' = G_v : H$ to denote that G' is a graph constructed by attaching the graph H on the graph G at v.

Proposition 2.9 (Colon Principle)**.** *In a* GREEN HACKENBUSH *game, any set of disjoint subgraphs connected to a common vertex can be replaced with a single bamboo stalk whose length is equal to the Sprague-Grundy value (i.e. taking the* NIM *sum of the individual Sprague-Grundy values) of the original subgraphs, without altering the game value.*

Proof. Denote $\mathcal{G}$ to be an arbitrary graph and denote x to be an arbitrary vertex in $\mathcal{G}$. Now, denote $\mathcal{T}_1$ and $\mathcal{T}_2$ to be two arbitrary trees that have the same Sprague-Grundy value. Denote $\mathcal{G}_1 = G_x : \mathcal{T}_1$ and $\mathcal{G}_2 = \mathcal{G}_x : \mathcal{T}_2$. We have that it suffices to prove that $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same Sprague-Grundy value. Hence, it suffices to show that the sum of $\mathcal{G}_1 + \mathcal{G}_2$ is a $\mathcal{P}-$position (as this will mean that $\mathcal{G}_1 + \mathcal{G}_2$ will have a Sprague-Grundy value of 0). Now, to show that the $\mathcal{G}_1 + \mathcal{G}_2$ is a $\mathcal{P}-$position, we will give a strategy for the players to follow. We invite the mimicking strategy, talked about in the beginning of the paper, to appear again. Suppose Left starts. Then if Left deletes a edge in $\mathcal{G}$ of one game, then Right should delete the same edge of $\mathcal{G}$ in the other game. Now, if Left deletes an edge in $\mathcal{T}_1$ or $\mathcal{T}_2$, then the Sprague-Grundy Values will need to be set equal again, and hence Right must make a move in $\mathcal{T}_1$ or $\mathcal{T}_2$ such that their Sprague-Grundy value is the same. Hence, this means that the Right will always make the last move. Similarly, if Right starts, then Left will make the last move. Hence, $\mathcal{G}_1 + \mathcal{G}_2$ is a $\mathcal{P}-$position. ∎

Notice that nowhere in our proof we had that $\mathcal{T}_1$ and $\mathcal{T}_2$ had to be branches of a *tree*. Hence, we have that this principle works for any rooted graph (any may even have multiple roots on the ground).

Now, consider Figure 5 again, looking at the single tree, we can "prune" this tree into a single bamboo tree by using the colon principle. A step by step process is shown in Figure 6. Notice that the tree is equivalent to a bamboo stalk! Hence, if we have trees in our GREEN HACKENBUSH game, we can also win by what we have developed previously.

2.3. **The Jungle At Night.** After a day of exploration in the jungle, we cannot *just* see bamboo stalks or trees anymore, but rather, we see all sorts of general (green) objects. We can analyze generalized GREEN HACKENBUSH. We can do this by understanding the *Fusion Principle.*

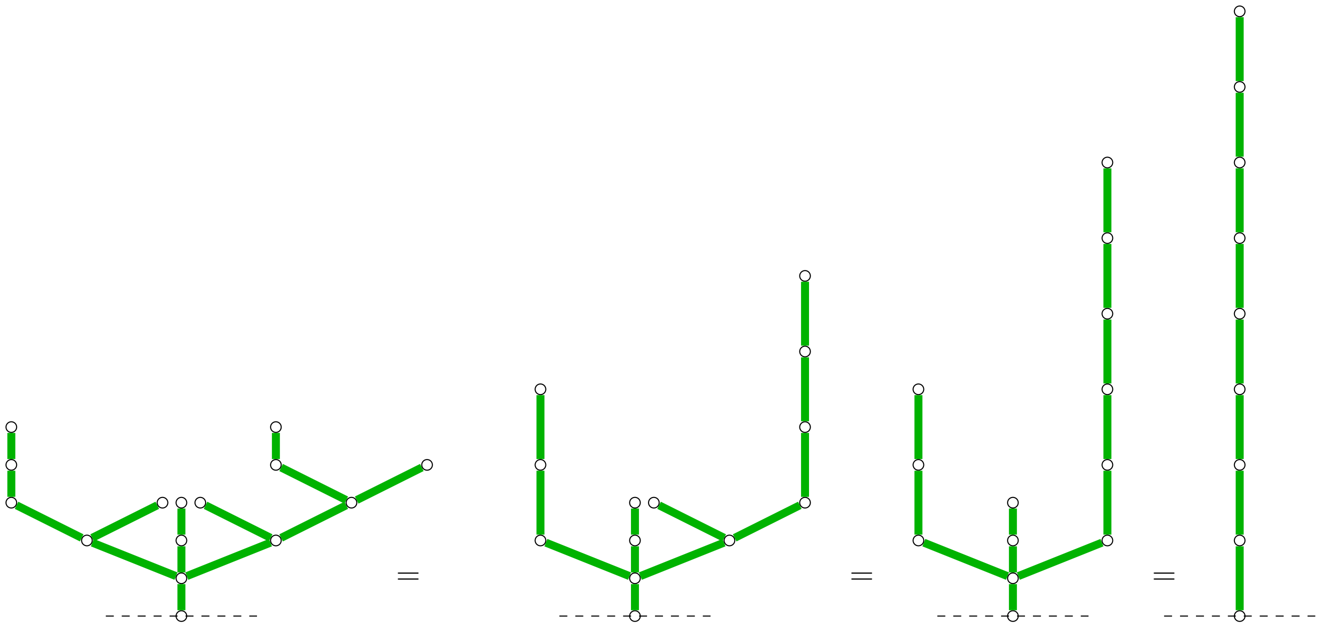

Figure 6. Colon Principle in Action on Figure 5

Proposition 2.10. *In any* GREEN HACKENBUSH *board, the value of the game does not change when fusing nodes together.*

Here, we have that a cycle (when the set of edges form a loop) can be simply thought as being replaced by an edge unattached at one end. Moreover, the process of fusing is by selecting two neighboring vertices and bringing them together to be a single vertex by bending the edge between them to form a loop. The proof of this principle can be found in [Bar06]; we omit the proof as it is not as beautiful and more of a casework, unlike the colon principle.

Notice that from the colon and fusion principles, we can always form an equivalent bamboo stalk of any GREEN HACKENBUSH position. Hence, we have that we can fully analyze any GREEN HACKENBUSH board and are always able to win (again, to re-emphasize this we are not able to win if the initial position is a $\mathcal{P}-$position).

3. Conclusion

After a long, tiring day, we were finally able to survive the green jungle! However, what if our original game board consists of red, blue, and green edges? Another journey awaits, but now climbing the purple mountain! Instead of seeing regular old bamboo stalks and trees, you will see colorful flowers (flower rule) and gardens (HACKENGARDEN)!

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