The Genus Theory of Misère Games and Genus Theory

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Abstract

This paper will examine the properties of misère games and how they play into genus theory. We will discuss the basic operations and structures behind misère games as well as further classifications of them. This paper heavily draws from Conway's *On Numbers and Games* and Siegel's *Combinatorial Game Theory*.

1 Introduction to Misère Games

In normal games, the convention is that the last person to make a move wins; in misère games, the last person to move loses. In this paper, we will use Aaron N. Siegel's notation meaning that we will denote the normal outcome of a normal game G as $o^+(G)$ and the outcome of the misère game G as $o^-(G)$. There will be other notation that we use that will be introduced later. Since the topic of this paper is about misère games, we will consider misère games to be the default in the sense that $o^-(G)$ will simply written as $o(G)$, and whenever we state $G = H$, we mean $G = H$ is misère play. If the need arises to distinguish between normal and misère play, then we will revert to the aforementioned notation. Now, let's explore misère nim.

2 Nim Games in Misère

In this section, we will lay out the groundwork of misère games and their connections to theorems and ideas in normal games. The information covered here will be invaluable when exploring the idea of Genus Theory. We will start by outlining some misère variations of theorems and definitions in normal play and then explore Nim arithmetic and values.

2.1 Misère Variations of Normal Theorems

[Con00] In normal games, Bouton's Theorem tells us that:

Theorem 2.1. *Nim position with heaps $a_1, a_2, a_3, ..., a_k$ is a $\mathcal{P}$-position in a normal game if and only if $a_1 \oplus a_2 \oplus a_3 \oplus ... \oplus a_k = 0$.*

The misère version of Bouton's Theorem is the same but it has a restriction:

Theorem 2.2. *Even if $a_1 \oplus a_2 \oplus a_3 \oplus ... \oplus a_k = 0$, the Nim position of G is not a misère $\mathcal{P}$ position if every $a_i = 0$ or 1. If this is true, then G can only be a misère $\mathcal{P}$ position if and only if $a_1 \oplus a_2 \oplus a_3 \oplus ... \oplus a_k = 1$.*

Definition 2.3. We define misère equality in impartial games the same way we do in normal games. Given two impartial games G and H and for all X, we can state that:

$$G = H \text{ if and only if } o^-(G + X) = o^-(H + X)$$

Misère games also have their own version of the Mex rule. In normal games, the Mex Rule states that:

Theorem 2.4. *If $G \cong *a_1, *a_2, *a_3, ..., *a_k$ and $a_1, a_2, a_3, ..., a_k \in \mathbb{N}$, then $G = *m$ is in normal play if $m = mex\{a_1, a_2, a_3, ..., a_k\}$*

The misère version of this rule is nearly the same, except that it has an additional condition at least one a_i is 0 or 1.

Let's state additional definitions and theorems before moving on to Nim arithmetic and values.

Theorem 2.5. *If $G \cong \{G'_1, G'_2, G'_3, ..., G'_k\}$ and $H \cong \{H'_1, H'_2, H'_3, ..., H'_k\}$ and $G'_i = H'_i \; \forall \; i$, then $G = H$.*

Definition 2.6. This definition is very important because it shows how simplification works in misère games. Let's say that:

$$G \cong \{G'_1, G'_2, G'_3, ..., G'_k\}$$

and that

$$H \cong \{H'_1, H'_2, H'_3, ..., H'_k, G'_1, G'_2, G'_3, ..., G'_k\}$$

Meaning that H is a game that contains the options of game G. We can say that, if the following conditions are true, then H can simplify to G:

- each new H'_j option has a reverting move, meaning that it includes a move back to G.
- if $G \cong 0$ then $o(H) = \mathcal{N}$

These extra constraints and rules for misère games in comparison to normal games is mainly due to how misère games treat 0, which is why so many misère properties resemble normal game properties, just with an extra condition.

Theorem 2.7. *If H simplifies to G, then $G = H$*

2.2 Nim Arithmetic and Values

Theorem 2.8. *For normal games, nim addition works as follows:*

Given some $a, b \in \mathbb{N}$, we can state that:

$$*a + *b = *c \text{ in normal play, when } c = a \oplus b$$

In misère games, nim addition works in the exact same way except it has an extra condition that at least one a or b is 0 or 1.

Proof: We can prove this theorem by applying the knowledge from section 2.1. Theorem 2.8 clearly holds when $a = 0$ or $b = 0$, so for our proof we will assume, without loss of generality, that $a > 0$ and $b = 1$, meaning that the options of $*a + *b$ are $*a + 0$ and $*a' + *(a' < a)$.

If we use induction on a, we can say that

$$*a' + * = *(a' \oplus 1)$$

for every such a'. This means that:

$$*a + *b = \{*a, *(0 \oplus 1), *(1 \oplus 1), ..., *((a-1) \oplus 1)\}$$

We can use the misère mex rule to show that:

$$a \oplus 1 = \text{mex}(\mathcal{A}) \text{ such that } \mathcal{A} = \{a, 0 \oplus 1, 1 \oplus 1, ..., (a-1) \oplus 1\}$$

Since $a \oplus 1 \notin \mathcal{A}$, we should show that $a' < a \oplus 1$ meaning that $a' \in \mathcal{A}$. If $a' = a$, then we have finished our proof. If $a' \oplus 1 < a$, it implies that $a' = (a' \oplus 1) \oplus 1 \in \mathcal{A}$. ■

Let us now examine misère nim values. In normal nim values, we define it recursively as:

$$\mathcal{G}^+(G) = \text{mex}_{G' \in G} \mathcal{G}^+ G'$$

In misère games, the nim value, where G is impartial, is defined as:

$$\mathcal{G}^-(G) = \begin{cases} 1 & \text{if } G \cong 0; \\ \text{mex}_{G' \in G} \mathcal{G}^-(G') & \text{otherwise .} \end{cases}$$

Some theorems relating to this which are essential to understanding Genus theory are:

Theorem 2.9. $\mathcal{G}^+(G + H) = \mathcal{G}^+(G) \oplus \mathcal{G}^+(H)\ \forall G, H$

Theorem 2.10. $\mathcal{G}^-(G + *) = \mathcal{G}^-(G) \oplus 1\ \forall G$

Theorem 2.11. *G is a game in nim position.*

- *If G has no heaps of size greater than or equal to 2, we can say that $G \cong n \cdot *$, then we can say that:*

$$\mathcal{G}^-(G) = \begin{cases} 1 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd .} \end{cases}$$

- *If G does have heaps of size greater than or equal to 2, then we can say that:*

$$\mathcal{G}^-(G) = \mathcal{G}^+(G)$$

.

Now, equipped with this knowledge, let us examine what the genus theory of misère games is.

3 Genus Theory of Misère Games

[Sie23]

3.1 Fickle and Firm Positions

Fickle positions have no heaps that are greater than or equal to 2 and firm positions have at least one such heap.

We have seen in the previous section that $\mathcal{G}^-(G+H)$ is not always equal to $\mathcal{G}^-(G) \oplus \mathcal{G}^-(H)$ even if they are the same Nim positions but Theorem 2.11 shows that, if the nim positions of G and H are both firm or fickle, then they are equal. Also, if we know what $\mathcal{G}^+$ and $\mathcal{G}^-$ are, then we can use the following conditions to see if G is firm or fickle:

Definition 3.1.

- If G is firm, then $\mathcal{G}^-(G) = \mathcal{G}^+(G)$
- If G is fickle then $\mathcal{G}^-(G) \neq \mathcal{G}^+(G)$

What this tells us is that if we are given the normal and misère nim values for G and H, then we can find $\mathcal{G}^+(G+H)$ and if $G+H$ is firm or fickle. Then, we can find $\mathcal{G}^-(G+H)$ and tells us $o^-(G+H)$. Therefore, if we combine the normal and misère nim values, we can get a new value that we call the genus of a game. The more rigorous definition of the genus of a game G is:

Definition 3.2. When $a = \mathcal{G}^+(G)$ and $b = \mathcal{G}^-(G)$, we write the genus of G as:

$$\mathcal{G}^{\pm}(G) = (a, b) = a^b$$

3.2 Tame and Wild Games

Example. $\mathcal{G}^{\pm}(*a) = \begin{cases} 0^1 & \text{if } a = 0; \\ 1^0 & \text{if } a = 1; \\ a^a & \text{if } a \geq 2. \end{cases}$

From this example, we can see that two genera (the plural of genus) are:

$$\mathcal{G}^{\pm}(*2 + *2) = 0^0$$

$$\mathcal{G}^{\pm}(*2 + *2 + *) = 1^1$$

These are the only genera in nim since Theorem 2.11 tells us that $\mathcal{G}^{\pm}(G) = 0^0$ or 1^0 when $G \cong n{\cdot}*$ (based on the parity of n) and because $\mathcal{G}^{\pm}(G) = a^a$ when G is firm where $a = \mathcal{G}^+(G)$

Games such as this are called tame games. A more rigorous definition of tame games is:

Definition 3.3. For every supposition H of game G, the game G is tame if:

$$\mathcal{G}^{\pm}(H) = a^a \text{ or } 1^0 \text{ or } 0^1 \text{ for some a.}$$

A wild game is any game that is not tame and therefore does not satisfy the above conditions.

An interesting outcome of this definition is that we can call a game G fickle if $\mathcal{G}^{\pm}(G) = 0^1$ or 1^0. If this equation does not hold true then we can call the game G firm. This outcome leads to a major theorem about tameness and fickleness.

Theorem 3.4. *$G + H$ is tame if G and H are also tame. $G + H$ is fickle if and only if G and H are both fickle as well.*

We will now define some more terms to understand tame games better:

Definition 3.5. A unit is defined as one of the following genera: $0^1, 1^1, 1^0, 0^0$

Definition 3.6. If all the options of game G are tame, then the discriminant of G is the set of units represented by G's options. We call the discriminant $\delta(G)$ and it is defined as:

$$\delta(G) = \{0^1, 1^1, 1^0, 0^0\} \cap \mathcal{G}^{\pm}[G]$$

where $\mathcal{G}^{\pm}[G] = \{\mathcal{G}^{\pm}(G') : G' \in G\}$

Using our new function $\delta(G)$, we can state this following theorem:

Theorem 3.7. *Assume that $G \not\cong 0$ and all G options are tame. Given this, we can say that:*
(a) G is wild if $\delta(G)$ has one fickle unit and one firm unit.
(b) G is tame and fickle if $\delta(G)$ has one fickle unit and no firm units.
(c) If $\delta(G)$ does not satisfy either of the conditions above, then G is tame and firm.

So, from what we have examined thus far, we can see that the ideal strategy for a general misère nim game is to play as you would in a normal nim game except for moves that would leave positions that have heap sizes of only 0 or 1. This is a general strategy, so given a game G such that $G = G_1 + G_2 + G_3 + ... + G_k$ where all G_i is tame, we can say that the ideal misère strategy is: play as you would in a normal game except when your moves leave positions that have fickle parts; if you run into such a case, then play until you leave an odd number of parts with the genus 1^0.

3.3 Restive and Restless Games

Here is a thought experiment: what would you call the genera of the discriminants of a game G where G is wild but G's options are tame?

The genera in this thought experiment belong to a whole new family of games, called restive and restless games.

Restive games have: $\delta(G) = \{0^1, 0^0\}$ and $\delta(G) = \{1^0, 1^1\}$ and have the following genera respectively: $1^2, 1^3, 1^4, ...$ and $0^2, 0^3, 0^4, ...$

Restless games have: $\delta(G) = \{0^1, 1^1\}$ and $\delta(G) = \{1^0, 0^0\}$ and have the following genera respectively: $2^0, 3^0, 4^0, ...$ and $2^1, 3^1, 4^1, ...$

Theorem 3.8. *G is a nim position and R is restive of genus a^b. Therefore, $R + G$ is a misère $\mathcal{P}$ if and only if $\mathcal{G}^+(G) = a$ except when every heap in game G has size $b, b \oplus 1, b, 0$, or 1. If G does have such heaps, then $R + G$ is a misère $\mathcal{P}$ position if and only if $\mathcal{G}^+(G) = b$*

Proof: We can prove this by using induction on game G. Notice that R, being restive of genus a^b, it has to have tame options for each of the genera:

$$(a \oplus 1)^0, (a \oplus 1)^1, 2^2, 3^3, ..., (b-1)^(b-1)$$

Given this, we can split the proof into three cases:

Case 1: Our goal here is to prove that $o(R+G) = \mathcal{N}$ in all cases where G has exactly one heap of size $m \notin \{b \oplus 1, b, 0, 1\}$ and all other heaps have size $b \oplus 1, b, 0, 1$.

In the subcase that $m > b$, m can move to $b \oplus 1, b, 0, 1$. One of these moves gives G' where $\mathcal{G}^+(G') = b$ such that $o(R+G') = \mathcal{P}$ using induction.

In the subcase that $m < b$ and game G has an odd number of heaps of size b or $b \oplus 1$, we can say that G has an option G' with $\mathcal{G}^+(G') = a$. Since G' has heaps of size m, we can say that $o(R+G') = \mathcal{P}$ by induction.

In the subcase that $m < b$ and game G has an even number of heaps of size b or $b \oplus 1$, we can say that $\mathcal{G}^+(G) = m$ or $m \oplus 1$. Because $b > m$ and $b > m \oplus 1$, we can see that R' has an option $\mathcal{G}^+(R') = \mathcal{G}^+(G)$. R' and G are tame and G is firm, meaning that we can say $\mathcal{G}^{\pm}(R'+G) = 0^0$

Case 2: Since every heap in game G has size $b, b \oplus 1, 0, 1$, we can say that $\mathcal{G}^+(G) = b, b \oplus 1, 0, 1$. We have to prove that $o(R+G) = \mathcal{P}$ when $\mathcal{G}^+(G) = b$ and is $\mathcal{N}$ in any other case.

In the subcase that $\mathcal{G}^+(G) = b \oplus 1$ and $b \oplus 1 > b$, then G has an option G' where $\mathcal{G}^+(G') = b$. Using induction, we can say that $o(R+G') = \mathcal{P}$, meaning that $o(R+G) = \mathcal{N}$.

In the subcase that $\mathcal{G}^+(G) = b \oplus 1$ and $b \oplus 1 < b$, then R must have an option R' where $\mathcal{G}^{\pm}(') = (b \oplus 1)^{b \oplus 1}$. R' is tame, so we have $\mathcal{G}^{\pm}(R'+G) = 0^0$, so $o(R'+G) = \mathcal{P}$.

In the subcase that $\mathcal{G}^{\pm}(G) = 0^1$ (respectively 1^0), we say that $o(R'+G) = \mathcal{P}$ where $\mathcal{G}^{\pm}(R') = (a \oplus 1)^0$ (respectively $(a \oplus 1)^1$).

In the subcase $\mathcal{G}^{\pm}(G) = 0^0$ or 1^1, we say that G is firm, so it has to have at least two heaps of size b or $b+1$. Each of these heaps can move to games 0 and $*$, so G has to be able to move to G' where $\mathcal{G}^{\pm}(G') = b$. This means that $o(R+G') = \mathcal{P}$ by induction.

In the subcase that $\mathcal{G}^+(G) = b$, we say that every G' has $\mathcal{G}^+(G') \neq b$ so $o(R+G') = \mathcal{N}$. We can also say that $R'+G$ is tame and firm, so $\mathcal{G}^{\pm}(R') \neq b^b$, meaning that $\mathcal{G}^{\pm}(R'+G) \neq 0^0$. This means that $o(R'+G) = \mathcal{N}$ and since $R+G \not\cong 0$, we can say that $o(R+G) = \mathcal{P}$.

Case 3: The game G has at least two heaps of a size that is not in $\{0, 1, b, b \oplus 1\}$.

In the subcase that $\mathcal{G}^+(G) = a$, we know that $\mathcal{G}^+(G) = a$, meaning that $\mathcal{G}^+(G') \neq a \ \forall G'$. By induction, $o(R+G') = \mathcal{N} \ \forall G'$. Since $\mathcal{G}^+(R) = a$, we have $\mathcal{G}^+(R') \neq a$ for any R'. This means that $\mathcal{G}^+(R'+G) \neq 0$ and since $R'+G$ is tame and firm, $o(R'+G) = \mathcal{N}$ The means that $o(R+G) = \mathcal{P}$

In the subcase that $\mathcal{G}^+(G) \neq a$, we say that if $a < \mathcal{G}^+(G)$, then $\mathcal{G}^+(G') = a$ for some G'. Using induction, we can see that $o(R+G') = \mathcal{P}$. However, if $a > \mathcal{G}^+(G)$, then we must have $a = 1$ and $\mathcal{G}^+(G) = 0$. Because we know that $a = 1$, we can say that $\delta(R) = \{0^1, 0^0\}$ so there is an option R' of genus 0^0. This means that $\mathcal{G}^{\pm}(R'+G) = 0^0$ which in turn means $o(R'+G) = \mathcal{P}$. No matter what the scenario is, $o(R+G) = \mathcal{N}$.

So, as seen from all three cases, Theorem 3.8 holds true. ■

4 Conclusion and Acknowledgements

There are many, many more applications in different areas of mathematics as well as in science that have not been discussed here. I would like to thank Simon Rubinstein-Salzedo and David Anthony Eun-Sung Walton for all the advice they have given me without which I could not have written this paper.

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