Types of Sums

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Preliminary Definitions

A game which has some sort of pieces arranged on a board is the configuration V_i. A *position* is denoted V_i^A or V_i^B. The V_i indicates the configuration, and the superscript indicates which of the players, A or B, will move next.

Definition 0.1. A *directed graph*, or digraph, is a graph in wherein the edges have a direction.

Each game G has a corresponding directed graph, denoted $\Gamma(G)$. The vertices of the graph are the positions of the game. The edges of the graph are the moves which can be under the rules of the game.

Definition 0.2. If there is an edge from V_a^A to V_b^B, V_b^B is called a follower of V_a^A. A position without followers is called *terminal*.

The class T of terminal positions has three outcome classes:

- T_A, a win for A,
- T_B, a win for B,
- T_O, a draw.

A vertex V_i^k which is such that plays starting from it have a finite maximum length, denoted $D(V_i^k)$, has a terminal distance of $D(V_i^k)$.

Conjunctive sum

If players A and B play simultaneously in a number of component games, $G_1, G_2, \ldots, G_n$, with each player making a move in some or all of the component games, this is a compound or composite game. In a conjunctive compound, the player makes a move in *every* component game. In a disjunctive compound, the player selects one of the games and moves in that, leaving all the other games unchanged. In a selective compound, the player chooses a set, which is not $\emptyset$, of component games. A move is made in each of them. More formally,

- In the disjunctive sum, denoted $G + H$, players move in exactly one of the two components.
- In the conjunction sum, denoted $G \wedge H$, players move in both components.
- In the selective sum, denoted $G \vee H$, players move in either or both components.

Definition 0.3. The *disjunctive sum* is defined as:

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}.$$

Definition 0.4. The *conjunctive sum* is defined as:

$$G \wedge H = \{G^L \wedge H^L | G^R \wedge H^R\}.$$

Definition 0.5. The *selective sum* is defined as:

$$G \vee H = \{G^L \vee H, G \vee H^L, G^L \vee H^L | G^R \vee H, G \vee H^R, G^R \vee H^R\}.$$

In general, when there are games with $k \geq 3$ components, like with

$$G_1 \wedge G_2 \wedge \cdots \wedge G_k,$$

a player needs to move in each of the k components. These sums can be considered within the context of two different ending conditions:

Definition 0.6. With the *short ending condition*, play is over when any of the components terminate.

Definition 0.7. With the *long ending condition*, play is over when all of the components terminate.

Let $C_1, C_2, \ldots, C_n$ denote a set of n uncolored directed graphs. The conjunctive compound of these graphs, C^{cnj} is defined in the following way: Let V_a^i be a vertex in C_i. The ordered set $V_p^{cnj} = \{V_a^1, V_b^2, \ldots, V_e^n\}$ is a vertex of C^{cnj}. The $V_a^1, V_b^2, \ldots, V_e^n$ are the *component* vertices of V_p^{cnj}.

The Remoteness Function

Informally, consider a game with $T = T_P$, where the last player wins. A position V_i^A would be considered a winning position for the next player A, if the player has a means of winning which will undoubtedly work no matter how the other player B chooses to move. As such, player A tries to win as quickly as possible. Conversely, the other player B tries to play as to delay defeat for the longest number of moves possible. This number of moves is called the remoteness $r(V_i^A)$ of V_i^A.

Definition 0.8. The *remoteness function* r describes how many moves the game will last if a player who can force a win tries to win as soon as possible and the losing player tries to lose as slow as possible.

As player A must move last to win, the remoteness of a position must be odd in this scenario.

Theorem 0.1. The remoteness value of a position V_i^k is given by:

- $r(V_i^k) = 0$ if V_i^k is a terminal position,
- $r(V_i^k) = 1 + r(V_{ij}^l)$ where $r(V_{ij}^l)$ is the least even number if V_i^k has at least one follower V_{ij}^l with $r(V_{ij}^l)$ even,

or

- $r(V_i^k) = 1 + r(V_{ij}^l)$ where $r(V_{ij}^l)$ is the greatest odd number.

Note that $L-$positions have even remoteness and $W-$ positions have odd remoteness, and $r(V_i^k) \geq 0$. When V_i^k is not terminal, $r(V_i^k) > 0$.

Theorem 0.2.

$$r(V_p^{cnj}) = \min[r(V_a^1), r(V_b^2), \ldots, r(V_e^n)] := r^*(V_p^{cnj}).$$

Lemma 0.1. There exists a follower $V_{p\pi}^{cnj}$ of V_p^{cnj} in the compound game such that $r^*(V_{p\pi}^{cnj}) = r(V_{c\gamma}^j)$

Proof. For each h,

$$r(V_d^h) \geq r^*(V_p^{cnj}) = r(V_c^j) > r(V_{c\gamma}^j).$$

From the definition of the remoteness function, there exists a follower $V_{d\delta}^h$ with

$$r(V_{d\delta}^h) \geq r(V_{c\gamma}^j).$$

The lemma follows by taking the j-th component of $V_{p\pi}^{cnj}$ to be $V_{c\gamma}^j$, and for each $h \neq j$, taking the $h-$th component to be $V_{d\delta}^h$. □

Lemma 0.2. Let V_{pq}^{cnj} be any follower of V_p^{cnj}. Then, if $r^*(V_{pq}^{cnj})$ is less than $r(V_{c\gamma}^j)$, it is odd.

Proof. Suppose that $r^*(V_{pq}^{cnj})$ is even and that

$$r^*(V_{pq}^{cnj}) < r(V_{c\gamma}^j).$$

Then, there is some component of V_{pq}^{cnj}, which will henceforth be called V_{fg}^k, for which

$$r^*(V_{pq}^{cnj}) = r(V_{fg}^k).$$

From this, it follows from the definition of $r*$ that

$$r^*(V_p^{cnj}) \leq r(V_f^k),$$

and from the definition of r that

$$r(V_f^k) \leq r(V_{fg}^k) + 1 = r^*(V_{pq}^{cnj}) + 1.$$

By supposition,

$$r^*(V_{pq}^{cnj}) + 1 < r(V_{c\gamma}) + 1.$$

By the definition of V_c^j,

$$r(V_{c\gamma}) + 1 \leq r(V_c^j) = r^*(V_p^{cnj}).$$

This is, however, a contradiction. □

Proof. Suppose, firstly, that $r^*(V_p^{cnj}) = r(V_c^j)$ is odd. Then there exists $V_{c\gamma}^j$ with

$$r(V_{c\gamma}^j) = r(V_c^j) - 1.$$

From the first lemma, there is therefore a $V_{p\pi}^{cnj}$ with $r*(V_{p\pi}^{cnj}) = r(V_p^{cnj}) - 1$, which is even. It follows from the second lemma that $r*(V_{pq}^{cnj})$ is the smallest which is even.

Suppose, now, that $r*(V_p^{cnj}) = r(V_c^j)$ is even. Then, if

$$V_{pq}^{cnj} = \{V_{aA}^1, \dots, V_{cC}, \dots, V_{eE}^n\}$$

is any follower of V_p^{cnj}, $r(V_{cC}^j$ is odd and less that $r(V_c^j)$. Thus,

$$r^*(V_{pq}^{cnj}) \le r(v_{cC}^j) < r(V_c^j) = r^*(V_p^{cnj})$$

and

$$r^*(V_p^{cnj}) \ge \text{super}_q r^*(V_{pq}^{cnj}).$$

However, the first lemma states that there is a $V_{p\pi}^{cnj}$ such that $r^*(V_{p\pi}^{cnj}) = r(V_{cC}^j)$. With that,

$$\begin{aligned}\text{super}_q r^*(V_{pq}^{cnj}) &\le \text{super}_\pi r^*(V_{pq}^{cnj})\\ &= \text{super}_C r(V_{cC}^j)\\ &= R(V_c^j) = r*(V_p^{cnj}).\end{aligned}$$

Finally, the goal is to show that $r^*(V_{pq}^{cnj})$ is odd for all q. Since $r(V_c^j)$ is even, the follower $V_{c\gamma}^j$ of V_c^j can be any of the followers V_{cC}^j. With that,

$$r*(V_{pq}^{cnj}) \le r(V_{cC}^j) = r(V_{c\gamma}^j),$$

which is odd. If there is strict equality, $r*(V_{pq}^{cnj})$ is odd. If there is strict inequality, $r^*(V_{pq}^{cnj})$ is odd by the second lemma. Therefore, r and $r*$ obey the same conditions and are therefore equal. □

Theorem 0.3. Given a game G that is the conjunctive sum of n games $G_1, G_2, \dots, G_n$, the position $x = [x_1, x_2, \dots, x_n]$ is losing if and only if

$$min(r(x_1), r(x_2), \dots, r(x_n))$$

is even.