The Angel and Devil Problem

Dhyutidhar Sravankumar

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1 Introduction

The Angel and Devil problem was a combinatorial game theory problem first proposed by John Conway. It is played on an infinite chessboard with 2 characters. The first is the Angel which can move p squares in any direction for an arbitrary power p that the Angel has. This leads to the Angel's reach being a square around the Angel with a side length of $2p + 1$. The second character in this game is the Devil. The Devil has an infinite range and can burn any square that it wants. Once a square is burnt by the Devil, the Angel cannot move there. The object of the game for the Devil is to trap the Angel such that the Angel cannot move to any position as they are all burnt. The object of the game for the Angel is to evade capture. This problem was solved in 3 dimensions for a while however that strategy could not be employed to prove the same in the 2 dimensional case. It has also been shown that an Angel of $p = 1$ loses to the Devil. The objective of the paper is to explain the proof of finding the lowest power which the Angel requires to win. It has been proven that $p = 2$ is the minimum power needed. In this paper we will explore one of the different solutions presented.

2 Observations

First one could imagine that an Angel which strictly travels in one direction can escape the Devil. In this case we can arbitrarily choose a strictly upward path. This kind of Angel is called a *Fool*. Although this idea looks intuitive it can easily be countered by the Devil. The Devil must build a wall at a certain range away from the current position of the *Fool*. The distance from the *Fool* can be characterized by the equation $d = 2^{4p^3}$. Then after the Angel has traveled $\frac{1}{2}$ of the distance from its current point to the wall, the Devil can focus in on the part that the Angel can travel to now. This strategy can beat the *Fool* for any 1 direction. However, what if the Angel wants to move strictly away from its origin. This way building only 1 wall would not work as the Angel could merely go in the other direction. We could employ a similar strategy however to the one in the beginning where we build 4 walls that entrap the Angel on 4 different sides. This strategy on its own cannot work as the Devil would not have enough

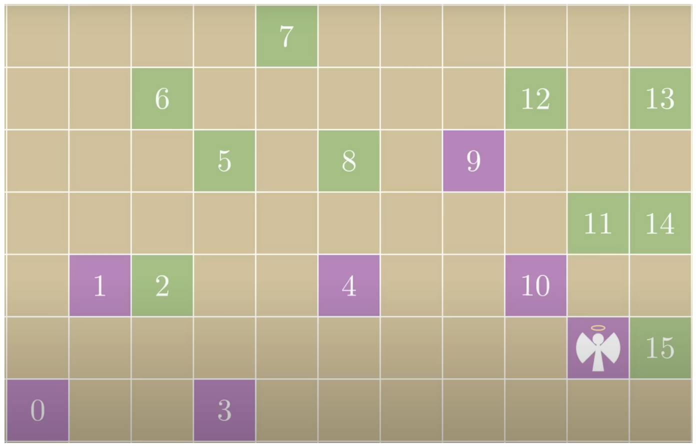

Figure 1: The Reduced Path [1]

time to trap the Angel. Rather the Devil must start building its wall 4 times further away. This leads to the equation $d = 4 \cdot 2^{4p^3}$ for the distance from the origin on each side. The next observation that must be presented is about the Angel. It must be observed that whenever the Angel moves, if the Angel moves to a square it previously moved to, then it has accomplished nothing while allowing the Devil to burn squares which benefit the Devil. This implies that any square that the Angel has previously moved to is theoretically burnt because moving there would result in worse play for the Angel. This means that every time the Angel makes a move it burns some squares on its own [1].

3 The Nice Devil

Definition 3.1: A *Nice Devil* is a type of Devil that will not get rid of a square that the Angel has moved to or had the opportunity to move to

Assuming that the Devil has a winning strategy, the Nice Devil begins by examining the path and finding a reduced path by finding the oldest square that the Angel can move to (Refer to Figure 2). It can then either do the exact move that the Devil would do or if it is not available, to pass the turn. It should also be noted that the Devil would not occupy one of the squares that the Angel had moved to in the scenario of the Nice Devil. This is because either the Nice Devil would have occupied it or the path reduction would have proved it wrong [2].

Theorem 3.2: As long as the Devil can trap the Angel with power p in some boundary $B(N)$ then the Nice Devil can do so as well.

4 The Runner

Definition 3.3: A *Runner* is a type of Angel that cannot move over burnt squares (The *Runner* can still squeeze through the diagonals of 2 squares)

The Runner will run around the burnt squares since it is not able to jump across them. It must also be noticed that the Runner is strictly weaker than the Angel since the Angel has all the same moves as the Runner plus the advantage of being able to jump over squares [2].

5 The Proof

Definition 5.1: A *journey* of an Angel with power p is a sequence of squares $(v_0, v_1, ..., v_n)$ such that $v_0 = (0, 0)$ and the $d(v_i, v_{i+1}) \leq p$ for each $0 \leq i \leq n$

Definition 5.2 : A set of squares S (of Z^2) is called *connected* if for each two distinct $a, b \in S$ there exists a path in S, $a = c_0, c_1, \ldots, c_n = b$ such that c_i and c_{i+1} are adjacent for each $0 \leq i < n$

Imagine that for every move that the Runner makes, it paints to the wall/burnt square nearest to it as seen in Figure 3. The only way for the Nice Devil to beat the Runner is to make the Runner loop back to its starting point by crossing the horizontal line in which it starts. However, this is an impossible task in optimal play because the maximum number of burnt squares s is $s \leq N + t$ where N is the number of squares in the original column and t is the number of turns. The edges painted is a maximum of $e_{painted} \geq 2t + 4 + 2N$. However, we also get the equation $s \leq N + t$. This would mean the Runner never reaches below the starting point making the Runner go up indefinitely. This would make the Nice Devil lose to the Runner of power 2 [1].

Theorem 5.3: The Runner of power 2 can defeat the *Nice Devil*

Theorem 5.4: The Angel of power 2 can defeat the *Nice Devil*

Theorem 5.5: The Angel of Power 2 can defeat the Devil

6 Closing Remarks

In this paper, we have examined the strategies in which we can prove how an Angel of power 2 can beat a Devil. We first examined the Nice Devil and its many movements. We then paired the Nice Devil up against the Runner, which was a much weaker opponent than the Angel. Lastly, we showed the final proof of how an Angel of power 2 wins. This proof is elegant in the form that we required not only the characters presented to us, rather we needed characters that we had to make ourselves to help solve this proof.

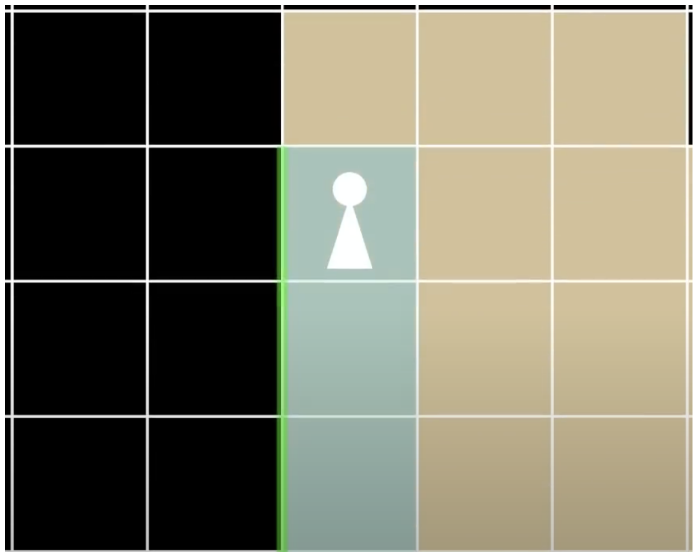

Figure 2: The Runner Painting a Wall [1]

7 References

[1] youtube.com/watch?v=JvhSRCfCHb4
[2] homepages.warwick.ac.uk/ masibe/angel-mathe.pdf
[3] youtube.com/watch?v=sxiKlOK3EJY