A Tour of the Surreal Numbers

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Abstract

In this paper, we will talk about surreal numbers. There's a lot to explore about them. We'll cover the basics of inequalities and arithmetic. We'll look at how ordinal numbers relate to surreal numbers, from recursion on ordinals to ordinal number arithmetic. Finally, we'll explore surreal number exponentiation and normal forms.

1 Introdouction

Surreal numbers are a relatively new discovery. They are said to contain "all numbers great and small": They contain the real numbers, infinite ordinals like ω, infinitesimals like $\frac{1}{\omega}$, and much more. The ordinal numbers have infinities that can grow arbitrarily big; for any set S of ordinals, no matter how huge the elements are, you can construct an ordinal bigger than everything in S. (Thus the collection of all ordinals forms a class, not a set.) The surreal numbers contain the ordinal numbers, but they extend them so that you can add, subtract, multiply, and divide in them, unlike in the ordinals.

The surreal numbers, denoted **No**, form an ordered field whose elements make up a proper class. A special property about the surreal numbers is that every ordered field whose elements form a *set* is contained in the surreal numbers! More precisely, every one of them is isomorphic to some ordered subfield of **No**. Thus the surreal numbers unify all these different number systems into one.

Another interesting thing is that they satisfy something close to algebraic closure, but not quite that. It's called *real closure*. The definition of real closure is a bit complicated and we won't get into that here, but proving it's true amounts to these two things: All nonnegative numbers in the field must have square roots, and all odd-degree polynomials over the field must have roots. This implies that, similar to extending $\mathbb{R}$ by the polynomial $x^2 + 1$ to get $\mathbb{C}$, taking $\mathbf{No}[x]/(x^2 + 1)$ gives an algebraically closed field just like $\mathbb{C}$. This field is called the *surcomplex* numbers.

The surreal numbers can be constructed from the following rules (see [2]):

- Every surreal number is created from a pair $\langle S_L \mid S_R \rangle$ of two sets of previously-created surreal numbers, such that no element of S_L is $\geq$ any element of S_R. The elements of S_L are called *left options* and the elements of S^R are called *right options*.
- For surreal numbers x and y, we say that $x \geq y$ (or $y \leq x$) if there's no right option x^R of x such that $x^R \leq y$, and there's no left option y^L of y such that $x \leq y^L$.
- We say that $x = y$ if $x \leq y$ and $y \leq x$.

We'll soon see how to make this a bit more precise.

Now, here's a question you probably have: If each surreal number is made from two sets of previously-created surreal numbers, then how do you even start? You'll always need previous surreal numbers to make new ones, so there's no possible way to create the 'first' one!

Well actually, you can create a surreal number with both sets empty: $\langle \{\} \mid \{\} \rangle$ (or $\langle \mid \rangle$). The empty set only contains surreal numbers, a vacuously true statement, so this is made up of only previous surreal numbers. And it's vacuously true that no element of the empty set is $\geq$ any element of the empty set, because it has no elements! Thus this is indeed a surreal number, the very first one. In fact, this is the number 0. Then there's $\langle \{0\} \mid \{\} \rangle = 1$ (or $\langle 0 \mid \rangle$) and $\langle \mid 0 \rangle = -1$. Again, the $\geq$ condition is vacuous. Through this process, you can keep making more and more numbers!

Now, how do we check when a surreal number is $\geq$ another based on other inequalities? Again, this comes down to a vacuous condition. We know that $0 \geq 0$: We need there to be no right option 0^R of 0 such that $0^R \leq 0$, and there to be no left option 0^L such that $0 \leq 0^L$. But these are both true since 0 has no left or right options! How about $1 \geq 0$? We need there to be no right option 1^R of 1 such that $1^R \leq 0$, and there to be no left option 0^L of 0 such that $1 \leq 0^L$. But there are no right options of 1 and no left options of 0, so this is vacuously true too!

What about checking whether $0 \geq 1$? We have to check that there's no right option 0^R of 0 such that $0^R \geq 1$, and that there's no left option 1^L of 1 such that $0 \geq 1^L$. Well, the first thing's vacuously true, but as for the second thing, 1 does have a left option: 0. And $0 \geq 0$ as we've seen, so $0 \geq 1$ has been proven false.

2 Birthdays

The rules we've mentioned for creating surreal numbers are a bit imprecise. We define inequalities from other inequalities. We define what's a number and what isn't using inequalities, but inequlities are defined using numbers. It all seems very circular. However, there is a way of making this more rigorous using (a type of) recursion.

Consider $\langle | \rangle = 0$. It's created from nothing; it doesn't use any numbers in its two sets. Because of this, we say that 0 is *born on day* 0. Next, since $\langle 0 | \rangle = 1$ and $\langle | 0 \rangle = -1$ only use numbers that are born on day 0, we say that they're born on day 1, the day after 0. Anything that only uses $-1, 0$, and 1 is born on day 2, and so on.

But this presents a problem. It turns out that on each day n, only rational numbers of the form $\frac{a}{2^k}$ get created. The finite days form the *dyadic rationals*:

Definition 2.1 A *dyadic rational* is a rational number of the form $\frac{a}{2^k}$ for integers a, k.

However, notice how I said those are formed on the **finite** days. Well, you can have days after that! The day that comes right after all the finite days is called day ω. On this day, we can construct the rest of the real numbers similar to the Dedekind cut construction: Each real number has the dyadic rationals below it in the left set, and the dyadic rationals above it in the right set. So the dyadic rational numbers are born throughout the finite days, and then on day ω suddenly all the rest of the real numbers are created! Day ω is referred to as the *big bang*.

There are a few extra day-ω surreal numbers:

$$\langle \mathbb{N} | \rangle = \omega;$$

$$\langle | -\mathbb{N} \rangle = -\omega;$$

$$\left\langle 0 \mid \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\} \right\rangle = \frac{1}{\omega},$$

and in general $d \pm \frac{1}{\omega}$ for all dyadic rationals d are included. Note that the *surreal number* ω is different from *day* ω.

But there's no need to stop at day ω! The day after ω is called day $\omega + 1$. Born on it are things like $\omega + 1$, $\omega - 1$, $\langle 0 \mid \frac{1}{\omega} \rangle = \frac{1}{2\omega}$, and more. The next day is day $\omega + 2$, and then $\omega + 3$, $\omega + 4$, etc.

There's even days after all these! The next one's day $\omega + \omega = \omega 2$. Then come $\omega 2 + 1$, $\omega 2 + 2$,..., and after all *those*, $\omega 3$. After all these expressions $\omega n + m$ is $\omega^2 = \omega \cdot \omega$. There's even ω^3, ω^4, ω^ω, ω^{ω^ω}, and even stuff after all the ω power-towers!

So what are these numbers? They're called the *ordinal numbers*. The idea is that for any *set* of ordinal numbers, you can always create ordinal numbers greater than everything in the set. You say that an ordinal number α is *greater than* an ordinal number β if it 'comes after it'. The way you use induction on ordinal numbers is to show that if something's true for all $x < \alpha$, it's true for α. (Note that there's no base case because, like we've seen before, it comes down to a vacuous condition.) We won't get into the precise definition of ordinal numbers, but this is the basic idea.

So instead of using only the nautral numbers for days, we use the ordinal numbers. The surreal numbers born on some day α are the numbers (not born earlier) whose left and right options are born on days $x < \alpha$. For instance, day $\omega + 2$ contains all numbers (not born earlier) whose options are born on finite days, day ω, and day $\omega + 1$. This is how we do recursion on ordinal numbers.

Next, $\leq$ and $=$ in the surreal numbers also use ordinal recursion in their definitions. Assuming numbers and inequalities are already defined for the days before day α, we know what constructions on day α give numbers and what don't, based on inequlities from previous days. Then, we can use our surreal number rules to define $\leq$ and $=$ between numbers born on days $\leq \alpha$.

The recursion for that is a bit complicated: To define each new condition $x^R \leq y$ and $x \leq y^L$, you have to dig deeper into, say, $x^R \leq y$, giving $x^{RR} \leq y$ and $x^R \leq y^L$ to define. But then $x^{RR} \leq y$ goes deeper again, and it keeps going all the way down. You have to do further recursion in the middle of each recursion step! However, despite the complexity, this construction is well-defined and a lot more rigorous.

3 Surreal Number Arithmetic

Now that we've defined the surreal numbers and inequalities, we can define addition, negation, and multiplication of surreal numbers. Note that we will say things like "$x = \langle x^L \mid x^R \rangle$" to mean that x^L spans over the left options of x and x^R spans over right options. When we say something like "$\langle y + x^L \mid y + x^R \rangle$," we mean all values $y + x^L$ for x^L in the left set of x (and similarly for $y + x^R$). Here are the definitions:

Let $x = \langle x^L \mid x^R \rangle$ and $y = \langle y^L \mid y^R \rangle$:

Definition 3.1

$$x + y = \langle x^L + y, x + y^L \mid x^R + y, x + y^R \rangle.$$

Definition 3.2

$$-x = \langle -x^R \mid -x^L \rangle.$$

Note the reversal of order of x^L and x^R.

Definition 3.3

$$x \cdot y = \langle x^L y + x y^L - x^L y^L,\ x^R y + x y^R - x^R y^R \mid x^L y + x y^R - x^L y^R,\ x^R y + x y^L - x^R y^L \rangle.$$

Note that these are all defined recursively, with addition and negation used in the definition of multiplication. (The definition also uses additive associativity.)

Now, it must be proven that these are well-defined: What if different *forms*, $\langle x^L \mid x^R \rangle$, for x and y give different results? And what if the result involves left and right options s^L, s^R such that $s^L \geq s^R$, making it not a number? Fortunately, it can be checked that these things don't happen, meaning that $+$ and $\cdot$ are well-defined. It also needs to be checked that if $a = a'$, $b = b'$, and $a \geq b$, then $a' \geq b'$, so that $\geq$ is well-defined too.

Also, it's been proven that the surreal numbers under addition, multiplication, and $\leq$ form an ordered field. And as the notation suggests, 0 is the additive identity, 1 the multiplicative, and $-x$ is the additive inverse of

x. Let's prove some of these! To start:

Theorem 3.4 $x + 0 = x$, $-0 = 0$, and $x \cdot 0 = 0$.

Proof. We prove these by induction using the definitions:

$$x + 0 = \langle x^L + 0, x + 0^L \mid x^R + 0, x + 0^R \rangle = \langle x^L + 0 \mid x^R + 0 \rangle,$$

since there are no 0^L or 0^R. Using the induction hypothesis, this simplifies to $\langle x^L \mid x^R \rangle = x$, as desired.

The next one is easy:

$$-0 = \langle -0^R \mid -0^L \rangle = \langle | \rangle = 0.$$

Now, it might look like $x \cdot 0 = 0$ will be the hardest one because the definition of multiplication is so complicated. However, notice that all the options of $x \cdot y$ involve some y^L's and y^R's. So, plugging in $y = 0$, we see that none of those expressions are possible since 0 has no left or right options. Thus $x \cdot 0$ simplifies to $\langle | \rangle = 0$.□

We use all these to prove the next thing:

Theorem 3.5 $x \cdot 1 = x$ (recall that $1 = \langle 0 \mid \rangle$).

Let's use induction and compute each of the terms. One of them is $x^L 1 + x 1^L - x^L 1^L$. Since the only 1^L is 0, this simplifies to

$$x^L 1 + x0 - x^L 0 = x^L 1 + 0 - 0 = x^L 1 + 0 + 0 = x^L 1.$$

And of course, by the induction hypothesis, $x^L 1 = x^L$.

Next is $x^R 1 + x 1^R - x^R 1^R$. Wait a minute! There are no 1^R's, so there can't be any instances of this! Thus the left options simplify to the x^L's. Similar computations show that $x^L 1 + x 1^R - x^L 1^R$ don't exist and $x^R 1 + x 1^L - x^R 1^L = x^R$. Therefore the right options simplify to the x^R's. Thus we get that

$$x \cdot 1 = \langle x^L \mid x^R \rangle = x,$$

as desired.□

Note that, although there are multiplicative inverses in the surreal numbers, their construction is very complicated and won't be talked about here.

4 More on Ordinal Numbers

Let's talk some more about ordinal numbers. Basically, they're a *well-ordered class* that extends the well-ordered *set* of nautral numbers:

Definition 4.1 A *well-ordered set* is a totally-ordered set $(S, \leq)$ such that any nonempty subset $U \subseteq S$ contains a smallest element: An element u such that $u \leq x$ for all $x \in U$. (And well-ordered *classes* are similar.)

We won't get into the definition of a *totally-ordered set,* because that would be off topic; basically it's a set equipped with a relation $\leq$ that satisfies the usual properties of inequalities. For example, the nautral numbers, integers, rationals, and reals are all totally-ordered sets. However, $\mathbb{R}$ isn't a *well*-ordered set because it has no smallest element; for every real number, you can subtract 1 to get a smaller (more negative) real number. The set $[0, \infty)$ of nonnegative real numbers still isn't a well-ordered set; even though it has a smallest element, 0, the subset $(0, \infty)$ doesn't. But the nautral numbers, $\mathbb{N}$, is a well-ordered set because every subset does have a smallest element. The ordinals are a well-ordered set extending $\mathbb{N}$.

The ordinal numbers can be constructed as sets of previous ordinal numbers. To start, $0 = \{\}$ because there's nothing before it. Then $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, etc. After that, $\omega = \mathbb{N}$, $\omega + 1 = \mathbb{N} \cup \{\omega\}$,

$\omega + 2 = \mathbb{N} \cup \{\omega, \omega + 1\}$, and so on forever. This is the typical construction used for the ordinal numbers.

Something interesting is that we can embed the ordinal numbers into the surreal numbers in the following way: Each ordinal number α is

$$\langle \{x < \alpha\} \mid \rangle,$$

where $\{x < \alpha\}$ means the set of embedded versions of the $x < \alpha$. So $0 = \langle \mid \rangle$, $1 = \langle 0 \mid \rangle$, $2 = \langle 0, 1 \mid \rangle$, etc. Then $\omega = \langle \mathbb{N} \mid \rangle$ as we've seen, $\omega + 1 = \langle \mathbb{N} \cup \{\omega\} \mid \rangle$, and so on for all the ordinals. Notice how they each have an empty set of right options.

So this embeds the ordinals and their inequalities into the surreal numbers. Although, this doesn't embed ordinal addition and multiplication; surreal addition and multiplication of ordinal numbers is different from their ordinal sum and product. Why is that?

Well, here's the definition of ordinal addition:

Definition 4.2 If you have two ordinal numbers α and β, you define their *ordinal sum* $\alpha + \beta$ using recursion on β:

$$\alpha + \beta = \alpha \cup \{\alpha + x : x < \beta\},$$

treating α as a set and assuming x is an ordinal.

As you might hope, the ordinal sum applied to nautral numbers is their regular sum.

Example 4.3 We have $\alpha + 0 = \alpha$ for all α, because $\alpha + x$ for $x < 0$ don't exist. If we add $\omega + 1$ together, we get $\mathbb{N} \cup \{\omega + 0\}$, or $\mathbb{N} \cup \{\omega\}$. Thus this indeed gives the ordinal after ω, which is what we've been calling $\omega + 1$. What if we compute $1 + \omega$? Well, this gives

$$\{0\} \cup \{1 + n : n \in \mathbb{N}\},$$

where n could be 0. But the elements in these sets are just all the nautral numbers! So we get $1 + \omega = \mathbb{N} = \omega$.

Wait a minute! $1 + \omega$ and $\omega + 1$ give different answers! How can this be? Well, we're used to addition being commutative: $a + b = b + a$ for all a, b. However, our example shows that ordinal addition isn't commutative. But it is still associative.

There's also ordinal multiplication:

Definition 4.4 You define $\alpha \cdot \beta$ again using recursion in β: It's the smallest ordinal greater than or equal to $\alpha \cdot x + \alpha$ for all $x < \beta$.

Example 4.5 Let's compute $\alpha \cdot 0$. Well, since ordinals $\alpha \cdot x + \alpha$ for $x < 0$ don't exist, it's vacuously true that every ordinal is greater than or equal to all of them. Thus the smallest such ordinal is 0, and $\alpha \cdot 0 = 0$, as one would expect. How about $\alpha \cdot 1$? Well, it's the smallest ordinal greater than or equal to $\alpha \cdot 0 + \alpha = 0 + \alpha$. We haven't shown that $0 + \alpha = \alpha$, but it's an easy check using induction. So $\alpha \cdot 1$ is the smallest ordinal greater than or equal to α, which is of course α.

Just like addition, ordinal multiplication is non-commutative. An example of this is that $\omega \cdot 2 = \omega + \omega$, a lot larger than ω, but $2 \cdot \omega = \omega$. That's why we express $\omega + \omega$ as $\omega 2$ and not 2ω. However, again it's still associative. It also satisfies left distributivity over addition:

$$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma).$$

But it doesn't satisfy right distributivity: It's not generally true that

$$(\beta + \gamma) \cdot \alpha = (\beta \cdot \alpha) + (\gamma \cdot \alpha).$$

For example, $(1 + 1) \cdot \omega = 2 \cdot \omega = \omega$, but $(1 \cdot \omega) + (1 \cdot \omega) = \omega + \omega = \omega 2$.

So, ordinal arithmetic satisfies just *some* of the nice properties we like. Surreal number arithmetic applied to ordinals is much nicer. For example, $1 + \omega = \omega + 1$ in the surrreal numbers, and they're both the ordinal after ω. And, $2 \cdot \omega = \omega \cdot 2 = \omega + \omega$. It satisfies all the properties that you know and love about nautral number arithmetic.

5 Exponentiation and Normal Forms

Now let's take a look at *normal forms.* To start, we define ω^x:

Definition 5.1

$$\omega^x = \{0, r\omega^{x^L} \mid r\omega^{x^R}\},$$

where r ranges over the positive real numbers, x^L (resp. x^R) ranges over left (resp. right) options, and the options in ω^x involve all possible combinations of these.

Example 5.2 If we calculate ω^0, since there are no x^L or x^R, we're just left with $\{0 \mid\} = 1$. Next,

$$\omega^1 = \{0, r\omega^0 \mid\} = \{0, r \mid\} = \omega,$$

since r can be an arbitrarily big real number. Also,

$$\omega^{-1} = \{0 \mid r\} = \frac{1}{\omega},$$

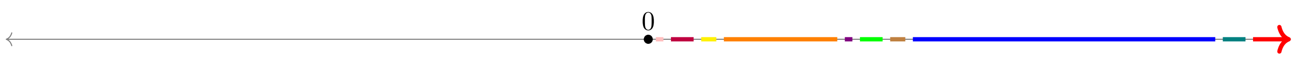
since r can be an arbitrarily small positive number.

As you can see, the reason the left option 0 is necessary, is that if there are no x^L, we need it to lower bound ω^x.

Now, what's the intuition behind this definition? Basically, if $\alpha > \beta$, then ω^α should be infinitely bigger than ω^β. So, it's bigger than $1000\omega^\beta$, or $1,000,000\omega^\beta$, or $r\omega^\beta$ for any positive real number r. They can't be *commensurate:*

Definition 5.3 For two positive surreal numbers x, y, we say that x is *commensurate with* y if there exists a positive integer n such that $x < ny$ and $y < nx$. (See [1])

Basically this is saying that x isn't infinitesimally smaller or infinitely bigger than y, instead they're *at the same scale,* so to speak. As intuition would suggest, commensurance is an equivalence relation whose equivalence classes are *convex* (i.e. if $x < z < y$ and x is commensurate with y, then x, y, and z are all commensurate). Here's a depiction of what that sort-of looks like:



As you can see, these commensurance classes are intervals (kind of) in the surreal number line. Because of their nature, there must be a simplest element in each one (we won't prove that here). We call these *leaders.* The mapping ω^x is obtained by letting ω^0 be the simplest of the leaders, 1, and letting ω^1 be the simplest leader to the right of that, ω^{-1} the simplest to the left of it, and so on; the structure of the surreal numbers applied to the leaders.

And indeed, exponentiation satisfies the properties we want:

Theorem 5.4

- $\omega^0 = 1$
- $\omega^{-x} = \frac{1}{\omega^x}$
- $\omega^{x+y} = \omega^x \cdot \omega^y$
- $\omega^1 = \omega$ (hence why we call it ω^x and not 2^x or e^x or something else.)

Everything is commensurate to a unique ω^x. That might make it seem like everything can be expressed as $r\omega^x$ for some real number r. But that's not quite true! Let's take a closer look.

Let x be a positive surreal number, and ω^{y_0} the unique leader commensurate with x. Then we can divide the real numbers into two classes, L for the real numbers t such that $\omega^{y_0} \cdot t \leq x$, and R for the t's such that $\omega^{y_0} \cdot t > x$. We have that L and R are nonempty, since there's a sufficiently large n such that $-n \in L$ and $n \in R$. We also have that everything in L is less than everything in R, and L and R are complements in $\mathbb{R}$. So, for reasons we won't get into here, exactly one of L or R has an extremal point (a maximum if L, a minimum if R). Call this point r_0, and write

$$x = \omega^{y_0} \cdot r_0 + x_1.$$

It follows that x_1 is *small compared to* x, i.e. nx_1 is between x and $-x$ for all integers n. If x_1 is zero, we're done. If x_1 is not 0, then we can construct in a similar way numbers r_1, y_1 such that

$$x_1 = \omega^{y_1} \cdot r_1 + x_2,$$

for x_2 small compared to x_1. (It could be that x_1 is negative, so we'd have to flip signs in the construction.) If again x_2 is non-zero, we can continue, producing the expansion

$$x = \omega^{y_0} \cdot r_0 + \omega^{y_1} \cdot r_1 + \cdots + \omega^{y_{n-1}} \cdot r_{n-1} + x_n$$

for each n. This terminates if $x_n = 0$ at any point. But usually these will not terminate for any n, so we have to keep continuing them throughout the infinite ordinals. There's a rigorous way this needs to be done, and we won't go into too much detail here, but these give unique expansions (called *normal forms*) for all x:

Theorem 5.5 For each x we can define a unique expression $\sum_{\beta<\alpha} \omega^{\gamma_\beta} \cdot r_\beta$ where α is some ordinal, the numbers r_β for $\beta < \alpha$ are non-zero reals, and the numbers γ_β form a decreasing sequence of numbers. These are distinct for distinct x, and every form satisfying these conditions occurs. (See [1])

Normal forms give us a nice way of identifying the *structure* of a number, so to speak. For instance, say you have a number

$$x = 2.5\omega^2 + \frac{4}{3}\omega^{0.0000043} + 56 + \frac{1}{3}\omega^{-100} + \frac{1}{242}\omega^{-1000.1} + \cdots + 2.01\omega^{-\omega}$$

(where $2.01\omega^{-\omega}$ is the ωth term). This normal form expansion shows that x is commensurate with ω^2, and that it's (relatively) infinitesimally close to $2.5\omega^2$. So that's the main term, our first approximation for x.

Next is the term $\frac{4}{3}\omega^{0.0000043}$. No matter how close $\omega^{0.0000043}$ might seem to $1 = \omega^0$, it's still infinitely bigger! In fact, even something like $\omega^{\frac{1}{\omega}}$ is infinitely bigger than ω^0. Any ω^α is as long as $\alpha > 0$. So the term $\frac{4}{3}\omega^{0.0000043}$ is the main term of $x - 2.5\omega^2$, and $2.5\omega^2 + \frac{4}{3}\omega^{0.0000043}$ is the second approximation for x. Then we keep adding terms, which make finer and finer changes as we go. At the very end, after all the finite-number-ith terms, is the ω-th term $2.01\omega^{-\omega}$. That's the tiniest, microscopic little term of them all! But it must be added to give the final result of x.

This is sort of analogous to decimal notation in the real numbers. With the different powers of 10 multiplied by single-digit numbers, the biggest term the most significant. This is like that but much more extreme, where the biggest term is infinitely bigger than the rest, ranging from huge, huge infinities to tiny, tiny infinitesimals.

As you can see, there's lots of interesting things to explore about the surrreal numbers. There are also more things we haven't covered, like the more general notion of *games*, the *omnific integers*, and so much more. The more we explore, the more new things we'll discover, and we're just getting started!

For more about the construction of the surreal numbers, see [2]. For more about normal forms, and about how surreal numbers tie into *games*, see [1].

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References

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