MANY-PLAYER NIM WITH THE PODIUM RULE

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1. Introduction

The normal Nim sum (for 2-player Nim) involves adding in base-2, where the addition in each column is done modulo 2. It's natural to ask whether the number 2 appears because there are 2 players (and 2 would become n if there were n players), or whether the appearance of 2 is inherent to Nim. In [1], Shuo-Yen Robert Li analyzed n-player Nim under the *podium rule*, yielding an elegant generalization of the Nim sum, and answering this question.

What is the podium rule? In many-player games, there can exist kingmaker scenarios, in which a player cannot win but can decide which of the other players will win. For example, suppose Alice, Bob, and Charlie play a game of Nim with one pile of two stones and one pile of one stone, with play proceeding in the order Alice - Bob - Charlie. Alice cannot win, because either Bob or Charlie will take the last stone. However, Alice gets to choose who wins. If she empties either pile, then Bob wins. If she takes one stone from the pile with two stones, then Charlie will take the last stone and win. The podium rule resolves such situations, and allows us to name a definitive winner.

2. The Podium Rule

Definition 2.1 (Podium rule)**.** At the end of a game, declare the first player who cannot make a move to be in last place, the player before them to be in first place (and the winner), and order each other player accordingly - the player before the first place player being in second place, the player before the second place player being in third place, and so on. Each player plays to minimize their podium position.

Example. Suppose there are three players; Alice, Bob, and Charlie; with play proceeding in that order. Then if Alice cannot win, she would prefer that Bob wins rather than Charlie. Similarly, if Bob cannot win he wants Charlie to win, and if Charlie cannot win he wants Alice to win.

Example. Suppose there are n players $1, 2, 3, \ldots n$, who play in that order. Then from player 1's point of view, the best outcome is player 1 winning, followed by player 2 winning, then player 3 winning, and so on.

The podium rule will be assumed from this point on.

Theorem 2.2. *Let G be a short impartial game, to be played with some arbitrary number of players. Under the podium rule, there is a unique winner of G.*

Proof. Induct on the birthday of G. If G has birthday 0, the current player cannot move, so the player before them wins. For an arbitrary game $G \neq 0$, each of its options has a unique winner by induction, and the player to move will choose an option that leads to their favorite winner. ■

Definition 2.3. If the previous player will win, we say a position is a 0-position, or that it has *rank* 0. (This is the same as a $\mathcal{P}$ position.) For $q > 0$, we say a position is an q-position, or has *rank* q, if the q-th player will win.

We can categorize the positions based on their options. If a player moves to a q-position, their podium position will be $q+1 \pmod{n}$ (since the next player will be in position q). So if every option of a position is an $(n-1)$-position, the position is a 0-position, and if the smallest rank of its options is q where $q < n-1$, then it is a $(q+1)$-position.

Lemma 2.4. *A position is a* 0*-position if and only if it cannot reach a* 0*-position within* $n-1$ *moves.*

Proof. Every move from a 0-position goes to a $(n-1)$-position. Each of the following $n-2$ moves decreases the rank of the position by at most 1. Thus a 0-position cannot reach another 0-position within $n-1$ moves. On the other hand, in a q-position for $0 < q \leq n-1$, there is a 0-position that can be reached in q moves, since each move can decrease the rank by 1. ■

From this, we derive a partition theorem.

Theorem 2.5. *Suppose* S *is a set of positions with the following properties:*

(1) *If* $G \in S$*, then* G *cannot reach another position in* S *within* $n-1$ *moves.*
(2) *If* $G \notin S$*, then* G *can reach some position in* S *within* $n-1$ *moves.*

Then S *is the set of* 0*-positions.*

Proof. By Lemma 2.1, the set of 0-positions has the desired property. We now show that the conditions on S uniquely determine its elements, so there is only one possible S. First, $0 \in S$, since 0 has no options that could possibly lead to a position in S. Now, induct on the birthday of G. Whether $G \in S$ or not is determined by whether any of its subpositions that can be reached in $n-1$ moves are in S. Each of these subpositions has smaller birthday, so this is known and thus we know whether $G \in S$ or not. ■

3. Analyzing Many-Player Nim

Definition 3.1. Let G be the Nim game with piles $c_1, c_2, \ldots c_k$, to be played with n players Let $\Delta(G)$ be the base-n number formed by converting each c_i to binary, and adding vertically, except that in each column we add modulo n.

When $n = 2$, this is the Nim sum of $c_1, c_2, \ldots c_k$.

Theorem 3.2. *Consider a Nim game* G *played with* n *players. This game is a* 0*-position if and only if* $\Delta(G) = 0$*.*

Proof. By Theorem 2.2, we need to prove two things. First, if $\Delta(G) = 0$, then for any subposition F of G reachable within $n-1$ moves, $\Delta(F) \neq 0$. Second, if $\Delta(G) \neq 0$, then there exists a subposition F of G reachable within $n-1$ moves, such that $\Delta(F) = 0$.

Start with the first part. Suppose $\Delta(G) = 0$. If every pile has size 0, then clearly G is a 0 position. If not, then when we add the piles in columns to compute $\Delta(G)$, any column containing some 1s contains n 1s. Any move in G will change some 1 to a 0, so that the sum in that 1's column is $n-1$. The following $n-2$ moves will reduce the sum in that column at most $n-2$, so after $n-1$ moves the sum in that column will be nonzero. Thus, if the resulting subposition is F, then $\Delta(F) \neq 0$ as desired.

Now, suppose that $\Delta(G) \neq 0$. Let the piles in G have sizes $c_1, c_2, \ldots c_k$. We want to show that, by replacing up to $n-1$ of these piles with smaller piles, we can reach a game F such that $\Delta(F) = 0$. Here's how. First, write each c_i in binary, arrange them vertically, and add modulo n in each column, as we do to compute $\Delta(G)$. Start in the leftmost column with a nonzero sum, and say that sum is $m \neq n-1$. Choose m rows with a 1 in that column, and for each such row, replace that 1 with a 0, and replace every digit to its right with a 1. Update the column sums, and repeat in the new leftmost column with a nonzero digit, modifying rows that have already been modified before modifying unmodified rows. Repeatedly applying this process will zero every column, and at most $n-1$ numbers will be changed in the process, since any previously modified row has a 1 in every future column of interest, so at no point will a row be modified after $n-1$ rows already have been. Each modified row has a smaller number than it had at the start, since the leftmost change in each row changes a 1 to a 0. Thus the resulting rows, read as binary numbers, give a subposition F of G reachable within $n-1$ moves such that $\Delta(F) = 0$. ■

The algorithm just described is rather complex, so let's see it in action.

Example. Take a 4-player Nim game G with piles of size $47, 4, 20, 23$, and 44. First, convert pile each to binary, arrange them vertically, and add modulo 4 in each column to compute $\Delta(G)$:

1	0	1	1	1	1
0	0	0	1	0	0
0	1	0	1	0	0
0	1	0	1	1	1
1	0	1	1	0	0
2	2	2	1	2	2

We find $\Delta(G) = 222122_4 \neq 0$. Now, let's see how we can reach a 0-position in 3 moves. In the first column, we have a sum of 2, so we change the two rows 101111 and 101100 to 011111. Let's update our table:

0	1	1	1	1	1
0	0	0	1	0	0
0	1	0	1	0	0
0	1	0	1	1	1
0	1	1	1	1	1
0	0	2	1	3	3

The third column now has a sum of 2. Since our construction puts 1s to the right of the first modified digit, the 2 rows we already modified have 1s in this column, so we just modify them again, now to 010111.

0	1	0	1	1	1
0	0	0	1	0	0
0	1	0	1	0	0
0	1	0	1	1	1
0	1	0	1	1	1
0	0	0	1	3	3

The fourth column now has a sum of 1, which we can fix by just changing either of the previously modified rows from 010111 to 010011.

$$\begin{array}{cccccc}
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 3 & 3
\end{array}$$

The fifth column now has a sum of 3, requiring us to modify the 3 rows with a 1 in this column. This includes the 2 already modified rows, so the total number of modified rows won't exceed 3.

$$\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 3
\end{array}$$

Lastly, to fix the sixth column, we remove a 1 in that column from each of the 3 modified rows. There are no digits to the right to change.

$$\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

And we're done! Comparing to the original piles, we see that a combination of three moves that leads to a 0-position is moving in 47 to 16, moving in 23 to 20, and moving in 44 to 20.

So, why is 2 such an important number in the Nim sum? Theorem 3.2 shows that the use of base-2 is intrinsic to Nim, but we only sum each column modulo 2 because there are 2 players. When there are n players, we add modulo n.

References

[1] S -Y R Li. "N-person nim and n-person moore's games". In: *International Journal of Game Theory* 7 (1978), pp. 31–36.