A WINNING STRATEGY IN FIBONACCI NIM

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1. Introduction

The game of Fibonacci nim consists of one pile of stones, where two players alternate removing stones from the pile, making it a *subtraction game*. Here, both players have the same moves available to them, so this is a *short impartial game*.

Suppose we have a pile of n stones. Then for the starting move, the first player can remove anywhere up to $n-1$ stones from the pile. For all subsequent moves, each player can remove anywhere up to twice as many stones as were removed by the previous player. Note that the players cannot pass on any given turn and must remove at least one stone from the pile. The player who removes the last stone wins the game. Let us look at an example.

Example 1.1. Consider the Fibonacci nim game with a pile of 12 stones, where we start first. Note that we can remove anywhere between 1 and 11 stones from the pile and, on the subsequent turn, our opponent can remove up to twice as much as we remove. Therefore, if we remove more than 3 stones, our opponent can take the rest and win.

Suppose we remove 1 stone, leaving 11 stones in the pile. This implies that our opponent can remove up to 2 stones. Suppose they remove 2 stones, leaving 9 stones in the pile. Now, we can remove up to 4 stones, but removing more than 2 stones will allow our opponent to win, so suppose we remove 1 stone again, leaving 8 stones in the pile.

In this case, our opponent can remove up to 2 stones. Suppose they remove 1 stone, leaving 7 stones in the pile. Then we can remove up to 2 stones. This time, we can remove 2 stones, leaving 5 stones in the pile. Here, our opponent can remove up to 4 stones from the pile, but removing 2 or more stones will allow us to win immediately. Therefore our opponent will remove 1 stone, leaving 4 stones in the pile.

Now, we can remove up to 2 stones, but removing 2 stones will result in our opponent winning, so we will remove 1 stone again, leaving 3 stones in the pile. In this situation, our opponent can remove either 1 or 2 stones, but in both cases, we will win.

Did we both play optimally or could our opponent play differently and win? What is the winning strategy? We will first define some notation. An $\mathcal{N}$ position is one in which the player who starts first wins the game. A $\mathcal{P}$ position is one in which the player who starts first loses the game. This would imply that when it is our turn to play, we would want to be in an $\mathcal{N}$ position, and we would want to make a move to a $\mathcal{P}$ position.

In this paper, we will study Fibonacci nim positions to determine which ones are $\mathcal{N}$ or $\mathcal{P}$ positions, and we will find a winning strategy. Surprisingly, it turns out that this requires understanding how every number can be expressed as the sum of distinct Fibonacci numbers, no two of which are consecutive. Given a game with one pile of size n, we will analyze the *Zeckendorf representation* of n [1] using the Fibonacci numbers that sum to n satisfying the properties above, and we will follow Whinihan [2] and Larsson and Rubinstein-Salzedo [3].

2. Zeckendorf representations

As we described earlier, Fibonacci numbers are key to understanding which positions are $\mathcal{N}$ or $\mathcal{P}$ positions. We note that we will index the Fibonacci numbers conventionally so that $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$ and so on. Let us define the Zeckendorf representation of a number.

Definition 2.1. A positive integer n is said to have a *Zeckendorf representation* if it can be expressed as the sum of distinct Fibonacci numbers, no two of which are consecutive.

For any n with $n = a_1 + a_2 + \ldots + a_k$, where all the a_is are Fibonacci numbers satisfying the property above, we can write the ith part of the Zeckendorf representation as $z_i(n) = a_i$. As an example, we can write 12 as $1 + 3 + 8$, 13 as itself (already a Fibonacci number), and 9 as $1 + 8$. This implies we get $z_1(12) = 1$, $z_2(12) = 3$, $z_3(12) = 8$, $z_1(13) = 13$, $z_1(9) = 1$, and $z_2(9) = 8$.

We can now state the following important theorem, which is key to understanding the $\mathcal{N}$ and $\mathcal{P}$ positions in Fibonacci nim.

Theorem 2.2. (Zeckendorf, 1972) [1] *Every positive integer has a unique Zeckendorf representation.*

3. A winning strategy

To study this game, we must consider all possible positions that can be reached at any point during the game. We can label each game position as (n, r) where n is the total number of stones in the position and r is the maximum number of stones the player can remove. We will prove that there is a specific criterion based on Zeckendorf representations that determines if a position is an $\mathcal{N}$ or $\mathcal{P}$ position. We now define safe and unsafe positions, as described in [2].

Definition 3.1. Let (n, r) be any position in a game G, and let $n = a_1 + a_2 + \ldots + a_k$ where all the a_is are distinct non-consecutive Fibonacci numbers. We say that (n, r) is an *unsafe position* if $r \geq z_1(n)$, and *safe position* if $r < z_1(n)$.

We will show that an unsafe position is an $\mathcal{N}$ position and a safe position is a $\mathcal{P}$ position by showing that unsafe and safe positions have all the properties of $\mathcal{N}$ and $\mathcal{P}$ positions respectively.

Theorem 3.2. *Let (n, r_m) be an unsafe position on the mth turn in a game G with $n = a_1 + a_2 + \ldots + a_k$, where all the a_is are distinct non-consecutive Fibonacci numbers. Then there exists a move to a safe position.*

Proof. Since (n, r_m) is an unsafe position, we know that the maximum number of stones that can be removed on the mth move satisfies $r_m \geq z_1(n)$. We also know that $z_2(n) = a_2$. We would like to show that the maximum number of stones r_{m+1} that can be removed on the $(m+1)$th turn is less than $z_1(n - x)$ where x is the number of stones removed on the mth turn. By definition, we know that $z_1(n)$ and $z_2(n)$ are not consecutive Fibonacci numbers, so let b be a Fibonacci number such that $z_1(n) < b < z_2(n)$. Then

$$2z_1(n) < b + z_1(n) \leq z_2(n),$$

since b and $z_2(n)$ are not necessarily consecutive. If $z_1(n)$ stones are removed on the mth turn, then for the $(m+1)$th turn, $r_{m+1} = 2z_1(n) < z_2(n)$. Since

$$z_1(n-x) = z_1(n - z_1(n)) = z_1(a_2 + a_3 + \ldots + a_k) = z_2(n),$$

then $r_{m+1} < z_1(n-x)$, so this is a safe position. □

We now prove the other direction that all moves from a safe position are to an unsafe position.

Theorem 3.3. *Let (n, r_m) be a safe position on the mth turn in a game G with $n = a_1 + a_2 + \ldots + a_k$, where all the a_is are distinct non-consecutive Fibonacci numbers. Then all moves are to an unsafe position.*

Proof. Since (n, r_m) is a safe position, we know that the maximum number of stones that can be removed on the mth move satisfies $r_m < z_1(n)$. Let x be the number of stones removed on the mth turn. Then by Zeckendorf's Theorem, we can write $z_1(n) - x$ as the sum of non-consecutive distinct Fibonacci numbers $f_{s_1} + f_{s_2} + \ldots + f_{s_j}$ where f_i represents the ith Fibonacci number ($f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3$, and so on) and $s_1, \ldots, s_j$ is a increasing sequence of natural numbers (f_{s_1} is the smallest). We note that $(n-x)$ can be written as

$$f_{s_1} + \ldots + f_{s_j} + z_2(n) + \ldots + z_k(n),$$

which is a sum of distinct non-consecutive Fibonacci numbers. We want to show that the maximum number of stones that can be removed on the $(m+1)$th move satisfies $r_{m+1} \geq z_1(n-x) = f_{s_1}$.

The largest possible value for the sum $f_{s_1} + f_{s_2} + \ldots + f_{s_j}$ that can be obtained is where f_{s_j} is the largest Fibonacci number less than $z_1(n)$, and $s_1, \ldots, s_j$ are spaced two apart ($s_2 - s_1 = \ldots = s_j - s_{j-1} = 2$). When this is true, then the sum can be evaluated as follows. When we add $f_{s_1} + f_{s_2} + \ldots + f_{s_j}$ to the Fibonacci number immediately preceding f_{s_1}, which we will call b_1, we see that $b_1 + f_{s_1}$ is equal to the Fibonacci number immediately preceding f_{s_2}, which we will call b_2. When we add b_2 to f_{s_2}, we get the Fibonacci number immediately preceding f_{s_3}, which we will call b_3. Continuing in the same manner, we can equate $b_1 + f_{s_1} + f_{s_2} + \ldots + f_{s_j}$ to $b_{s_j} + f_{s_j}$ to get $z_1(n)$. Since the terms in the sum can take on smaller values, we find that

$$b_1 + f_{s_1} + f_{s_2} + \ldots + f_{s_j} \leq z_1(n),$$

which implies

$$z_1(n) - (f_{s_1} + \ldots + f_{s_j}) = x \geq b_1,$$

and hence

$$r_{m+1} = 2x = 2(z_1(n) - (f_{s_1} + \ldots + f_{s_j})) \geq 2b_1.$$

If we let b_0 be the Fibonacci number immediately preceding b_1, then we see that

$$2b_1 \geq b_1 + b_0 \geq f_{s_1}.$$

Therefore $r_{m+1} \geq f_{s_1}$, so we have moved to an unsafe position. □

We can then use the Partition Theorem, which we will state below, to show that any unsafe position is an $\mathcal{N}$ position and any safe position is a $\mathcal{P}$ position.

Theorem 3.4 (Partition Theorem)**.** *Let S be a set consisting of short impartial games and all of their subpositions. Suppose S can be partitioned into the sets P and N so that $P \cap N = \emptyset$ and $P \cup N = S$, and that any game in N has a move to a $\mathcal{P}$ position and for any game in P, all moves are to an $\mathcal{N}$ position. Then N is the set of $\mathcal{N}$ positions and P is the set of $\mathcal{P}$ positions.*

Proof of Theorem 3.4. If N contains games that are in $\mathcal{P}$, then for these games, there are no moves to a $\mathcal{P}$ position which contradicts our definition of N. If N contains games that are in $\mathcal{N}$, then every game in N has a move to a $\mathcal{P}$ position. Similarly, if P contains games that are in $\mathcal{P}$, then for each of these games, every move is to an $\mathcal{N}$ position. If P contains games that are in $\mathcal{N}$, then these games contain moves that are not to an N position. Since P does not contain any $\mathcal{N}$ position games, they must all be contained in N. Similarly, since N does not contain any $\mathcal{P}$ position games, they must all be contained in P. Therefore P is the set of $\mathcal{P}$ positions and N is the set of $\mathcal{N}$ positions. □

Theorem 3.5. *Any unsafe position is an $\mathcal{N}$ position, and any safe position is a $\mathcal{P}$ position.*

Proof of Theorem 3.5. We will use the Partition Theorem. Let S be the set of subpositions of the game G, and S be partitioned into the sets N and P such that N is the set of unsafe positions and P is the set of safe positions. We can clearly see that N and P are disjoint sets. Since there always exists a move from an unsafe position to a safe position by Theorem 3.2, and every move in a safe position is to an unsafe position by Theorem 3.3, the sets N and P consist of $\mathcal{N}$ and $\mathcal{P}$ positions respectively. □

As we have showed in the proof above, unsafe positions are $\mathcal{N}$ positions and safe positions are $\mathcal{P}$ positions. This implies that a winning move would take us from an unsafe position to a safe position. We found one such move in our proof of Theorem 3.2. That is, on the mth move in a pile of n stones, by removing $z_1(n)$ stones, the smallest Fibonacci number in the sum $n = a_1 + \ldots + a_k$, we are left with a safe position, so this is a winning move.

Now that we have found a winning strategy, we can analyze our original example (1.1) in which we started with a pile of 12 stones. To use our winning strategy, we will look at the Zeckendorf representation of 12 which is $12 = 1 + 3 + 8$. Since $z_1(12) = 1$ is less than 12, which is the maximum number of stones that can be removed on the first turn, this is an $\mathcal{N}$ position, so the player who starts first will win. The winning move for us is to remove 1 stone, leaving a pile of 11 stones. The Zeckendorf representation of 11 is $11 = 3 + 8$.

Since our opponent can remove up to 2 stones, and $z_1(11) = 3$ is greater than 2, this is a $\mathcal{P}$ position. Regardless of what our opponent plays, as long as we play optimally, we will win. If our opponent removes 2 stones, then there are 9 stones remaining and we can take up to 4 stones. The Zeckendorf representation of 9 is $9 = 1 + 8$, and since $z_1(9) = 1$ is less than 4, this is an $\mathcal{N}$ position and we should remove 1 stone, leaving 8 in the pile. If we continue this winning strategy, we will find that the moves given in Example 1.1 are the optimal moves to play.

While the winning strategy that we described in Theorem 3.2 works, there are many other winning strategies for Fibonacci nim. A complete categorization of all the winning moves in Fibonacci nim can be found in Allen and Ponomarenko [4].

4. Generalizations of Fibonacci nim

An immediate generalization of Fibonacci nim is to include more piles in the game. When doing so, we will find that there are two ways to play this version of the game. The first is where we have a *local move dynamic*, and the rule of removing up to twice as many stones as the previous player applies independently to each individual pile on stones.

Example 4.1. If there is a pile A of 12 stones and a pile B of 7 stones in a game with a local move dynamic, then if we play first and remove 2 stones from pile A, then our opponent can either remove up to 4 stones from pile A or up to 6 stones in pile B, since our opponent is the first to play in pile B.

The second way to play this game is where we have a *global move dynamic*, where the rule of removing up to twice as many stones as the previous player applies all moves regardless of which pile you are playing in.

Example 4.2. Consider the same piles A and B as in Example 4.1. Then if we start first and remove 2 stones as before, then our opponent must remove up to 4 stones from either pile A or B.

The local move dynamic version of Fibonacci nim is analyzed in [3] using Sprague-Grundy theory. The version of the game for a global move dynamic is analyzed in [5] where the $\mathcal{N}$ and $\mathcal{P}$ positions of the two-pile game are found using the Fibonacci word, a string of binary digits. It still remains not completely understood as to what the $\mathcal{N}$ and $\mathcal{P}$ positions are in any multi-pile Fibonacci nim game with more than two piles and a global move dynamic.

References

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