Scoring Games

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Our research in combinatorial game theory has largely focused on games in which the winner is determined by which player moves last. For example, in a NIM game, the last player removing the last stone wins. However, some popular games like TABLETOP CHECKERS, a game in which a player moves his pawn to move across the other's pawn and gets a point, do not fall into categories we studied. In this paper, we would describe the basic rules and definitions in genescoring game, a well-tempered scoring game and discuss the optimal.

1 Basic definitions in scoring games

To simplify the game, we assume that a scoring game ends when both players cannot move anymore, and the Left and the Right move alternately. At the end of the game, which player owns more points wins. Thus, the game we study can be defined as follows:

Definition 1.1 A *scoring game* G is either a real number (in which case we say G is *atomic*) or an ordered pair $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ and $\mathcal{R}$ are finite *nonempty* sets of scoring games. If G is is non-atomic, we refer to the elements of $\mathcal{L}$ and $\mathcal{R}$ as *left options* and *right options* of G, respectively. Otherwise we declare that G has no options. We use notation $\langle A, B, C, \cdots \mid D, E, F, \dots \rangle$ to denote the pair $(\{A, B, C, \dots\}, \{D, E, F, \dots\})$.

If the game G is non-atomic, Left can move from G to any left options and Right can move from G to any right options. If $G = x \in$ R is atomic, then the game ends with Left winning the game by x points. For example,

$$\langle 1 \mid -1 \rangle$$

is a game that ends after one move; whichever player moves first gains a point. It is quite similar to games we have learned before, but here, for scoring games, a certain game is ultimately a number. Because Left and Right move alternately, it is natural to apply Left stops and Right stops to quantify a game. Thus, we can define the outcome of a game as follows:

Definition 1.2 If G is a scoring game, the *left outcome* $LS(G)$ and the *right outcome* $RS(G)$ are the real numbers defined recursively as follows:

$$LS(G) = \begin{cases} G & \text{if G is atomic,} \\ max(RS(G^L) & \text{if not.} \end{cases}$$

$$RS(G) = \begin{cases} G & \text{if } G \text{ is atomic,} \\ min(RS(G^R)) & \text{if not.} \end{cases}$$

where G^L and G^R are variables ranging over the left and right options of G.
Thus, $LS(G)$ or $RS(G)$ is the final score of the game G under perfect play when the first player is Left or Right, respectively.

Definition 1.3 Addition and negation are defined as follows:

$$G+H=\begin{cases} G+H & \text{if } G \text{ and } H \text{ are atomic,} \\ \left\langle G^L+H, G+H^L \mid G^R+H, G+H^R \right\rangle & \text{otherwise.} \end{cases}$$

$$-G=\begin{cases} -G & \text{if } G \text{ is atomic,} \\ \left\langle -G^R \mid -G^L \right\rangle & \text{otherwise.} \end{cases}$$

Definition 1.4 Similarly, if G is non-atomic and $x \in \mathbb{R}$ is atomic, then x has no options, so

$$G+x=\left\langle G^L+x \mid G^R+x \right\rangle$$

Addition can be described as playing two games in parallel, and adding the final scores. Negation corresponds to reversing the roles of the two players. Addition of scoring games is associative and commutative, and the atomic game 0 is the additive identity.

So far, we constructed how to describe a scoring game. Then, it is time to tell $\mathcal{L}$, $\mathcal{R}$, $\mathcal{P}$, and $\mathcal{N}$ positions of a scoring game.

For $\mathcal{L}$ positions, Left will win no matter which player goes first. Thus, in this case $LS(G)$ and $RS(G)$ should all be positive; for $\mathcal{R}$ positions, both outcome should be negative. When it comes to $\mathcal{P}$ position, it becomes more complicated. $LS(G)$ should be negative, and $RS(G)$ should be positive. Also, for $\mathcal{N}$ positions, $LS(G)$ is positive, while $RS(G)$ is negative.

2 Well-tempered scoring games

2.1 Definitions

Based on the definitions of scoring games, we discuss operations of well-tempered scoring games. The temper means the length of the game. Well-tempered integer-valued scoring games have the property that the parity of the length of the game is independent of the line of play. In this part, we consider disjunctive sums of these games, and describe a theory analogous to the standard theory of disjunctive sums of normal-play partizan games.

We will generally refer to well-tempered scoring games simply as "$\mathbb{Z}$ -valued games" in what follows, and refer to the games of Conway's partizan game theory as "partizan games" when we need them. And like the way we denoted in hot games, we always simplify $\langle x \mid \langle y \mid z \rangle \rangle$ to $\langle x \mid\mid y \mid z \rangle$.

Definition 2.1 Let $\mathcal{S} \subseteq \mathbb{Z}$. Then, an even-tempered $\mathcal{S}$-valued game is either an element of $\mathcal{S}$ or a pair $\langle L \mid R \rangle$ where L and R are finite nonempty sets of odd-tempered S-valued games. An odd-tempered $\mathcal{S}$-valued game is a pair $\langle L \mid R \rangle$ where L and R are finite nonempty sets of even-tempered $\mathcal{S}$-valued games. A well-tempered $\mathcal{S}$-valued game is an even-tempered $\mathcal{S}$-valued game or an odd-tempered $\mathcal{S}$-valued game. The set of well-tempered $\mathcal{S}$-valued games is denoted $\mathcal{W}_\mathcal{S}$. The subsets of even-tempered and odd-tempered games are denoted ${\mathcal{W}_\mathcal{S}}^0$and ${\mathcal{W}_\mathcal{S}}^1$, respectively.

Next, we define outcomes of $\mathcal{S}$ class.

Definition 2.2 For $G \in \mathcal{W}_\mathcal{S}$, we define $L\left(G\right) = R\left(G\right) = n$ if $G = n \in \mathcal{S}$, and otherwise, if $G = \langle L_1, L_2, ... \mid R_1, R_2, ... \rangle$, then we define

$$L(G) = max\left\{R(L_1), R(L_2), ...\right\},$$
$$R(G) = min\left\{L(R_1), L(R_2), ...\right\}.$$

For any $G \in \mathcal{W}_\mathcal{S}$, $L(G)$ is called the left outcome of G, $R(G)$ is called the right outcome of G, and the ordered pair $(L(G), R(G))$ is called the (full) outcome of G, denoted $o^\sharp(G)$. In scoring games, as

Definition 1.2 states, the full outcome of G is the final score of G.

Another important fact that we need later is the following:

Proposition 2.3 If G is a $\mathbb{Z}$-valued game and n is a number, then

$$L(G+n) = L(G) + n$$

and

$$R(G+n) = R(G) + n$$

Proof. If G is a number, then, this is obvious; if G = $\langle L_1, L_2, \cdots \mid R_1, R_2, \ldots \rangle$, then n has no options, so

$$G + n = \langle L_1 + n, L_2 + n, \cdots \mid R_1 + n, R_2 + n, \ldots \rangle \, .$$

Thus, by **Definition 2.2** $L(G{+}n) = max \, \{R(L_1+n), R(L_2+n), ...\} = max \, \{R(L_1)+n, R(L_2)+n, ...\} = L(G)+n$, similarly, R(G+n) = R(G) + n. ■

In the theorem of impartial games, if game $G \geq 0$ and game $H \geq 0$, then the combination $G + H \geq 0$. Although partizan games are opposite to impartial games, for this rule, they are almost the same. Similarly, for $\mathbb{Z}$-valued games, we state the following theorem:

Theorem 2.3 Let G and H be $\mathbb{Z}$-valued games. If G and H are both even-tempered, then

$$R(G+H) \geq R(G) + R(H) \tag{1}$$

Likewise, if G is odd-tempered and H is even-tempered, then

$$L(G+H) \geq L(G) + R(H) \tag{2}$$

Proof. If G and H are both even-tempered, then (1) follows from Proposition 2.3 whenever G or H is a number, so suppose both are not numbers. Then every right option of G + H is either of the form $G^R + H$ or $G + H^R$. Since G^R is odd-tempered, by induction (2) tells us that $L(G^R + H) \geq L(G^R) + R(H)$. Clearly $L(G^R) + R(H) \geq R(G) + R(H)$, because R(G) is the minimum value of $L(G^R)$. So $L(G^R + H)$ is always at least R(G)+R(H). Similarly, $L(G + H^R)$ is always at least R(G) + R(H). So every right option of G + H has left outcome at least R(G) + R(H), and so the best right can do with G + H is R(G) + R(H), proving (1).

If G is odd-tempered and H is even-tempered, then G is not a number so there is some left option G^L with $L(G) = R(G^R)$. Then by induction, (1) gives

$$R(G^R + H) \geq R(G^R) + R(H) = L(G) + R(H). \tag{3}$$

Thus, this proves (2). ■

2.2 Operations

Unfortunately, some of the results above are contingent on parity. Without the conditions on parity, equations (1-3) would fail. For instance, if G is the even-tempered game $\langle -1 \mid -1 \mid\mid 1 \mid 1 \rangle$ and H is the odd-tempered game $\langle G \mid G \rangle$, then the player can easily check that R(G) = 1, R(H) = 1, but R(G + H) = 2 (Right moves from H to G), and -(2) $\ngeq$ 1 + (-1) , so that (1) fails. The problem here is that since H is odd-tempered, Right can end up making the last move in H, and then Left is forced to move in G, breaking her strategy of only playing responsively.

In order to avoid of unfortunate results, we consider a restricted class of games, namely i-game:

Definition 2.4 An i-game is a $\mathbb{Z}$-valued game G which has the property that every option is an i-game, ad if G is even-tempered, then $L(G) \geq R(G)$.

e.g. numbers are always i-games, $*$ and $\langle 1 \mid -1 \rangle$ are i-games. However, the game G = $\langle 1 + * \mid -1 + * \rangle$ is not. This is because it is even-tempered and $L(G) = -1 \leq 1 = R(G)$.

Then, similar to **Definition 2.2**, the theorem of i-game can be generalized to the following:

Theorem 2.5 Let G and H be $\mathbb{Z}$-valued games, and G an i-game.
• If G is even-tempered and H is odd-tempered, then

$$R(G + H) \geq R(G) + R(H) \tag{4}$$

• If G and H are both odd-tempered, then

$$L(G + H) \geq L(G) + R(H) \tag{5}$$

• If G and H are both odd-tempered, then

$$L(G + H) \geq R(G) + L(H) \tag{6}$$

Proof. The proof of **Theorem 2.5** is similar to that of **Theorem 2.3**. Note only difference is that **Theorem 2.5** list the condition of (5). If G and H are both even-tempered, then every equation follows from **Proposition 2.3** whenever G or H is a number, so suppose both are not numbers. To see (4), note that every right option of G + H is either of the form $G^R + H$ or $G + H^R$. Since $L(G^R) \geq R(G)$ and G^R is an odd-tempered i-game, (5) tells us inductively that

$$L(G^R + H) \geq L(G^R) + R(H) \geq R(G) + R(H).$$

And likewise since $L(H^R) \geq R(H)$ and H^R is even-tempered, (6) tells inductively that

$$L(G + H^R) \geq L(H^R) + R(G) \geq R(G) + R(H).$$

So no matter how Right moves in G + H, he produces a position with left outcome at least R(G) + R(H). This establishes (4). Equations (5-6) can easily be seen by having Left make an optimal move in G or H, respectively, and using (4) inductively. ■

For the outcome of an i-game, we have these definitions:
Theorem 2.6 If G is an even-tempered i-game, and $R(G) \geq 0$, then for any $X \in \mathcal{W}_{\mathcal{Z}}$, we have $o^\sharp(G + X) \geq o^\sharp(X)$.

Proof. If X is even-tempered, then by Equation (1),

$$R(X) \leq R(G) + R(X) \leq R(G + X),$$

and by Equation (6) we have

$$L(X) \leq R(G) + L(X) \leq L(G + X),$$

If X is odd-tempered, then by Equation (4),

$$R(X) \leq R(G) + R(X) \leq R(G + X),$$

and by Equation (2) we have

$$L(X) \leq R(G) + L(X) \leq L(G + X),$$

Thus, we can deduce that, generally, $o^\sharp(G + X) \geq o^\sharp(X)$. ■

3 Dots and Boxes

3.1 Game Rules

General CGT cannot be directly applied to Dots and Boxes, because it is a scoring game, which means that the winner is not necessarily the player of the last move. If we just replace the winning condition with the normal ending condition, then we get a game that can be analyzed with this theory, which is actually very related to the original game. In this part, we introduce the rules and briefly show how the Dots and Boxes game is played (mainly focused on the endgame by Right).

First, the rules of Dots and Boxes are very simple:

i) Start with a rectangular dots grid.
ii) On his turn each player draws a horizontal or vertical segment joining two (not previously joined) adjacent dots.
iii) If a player draws the fourth side of a square ("box") then he claims it and must make another move. Notice that several captures can be made in a single turn.
iv) When all the boxes have been claimed, the player who owns the most is the winner.

Then, let us see how a typical Dots and Boxes game can be played. Assume that both players avoid to draw the third edge of a box while possible, so that the opponent cannot capture any boxes. In this case, we reach an endgame position like that of Figure 4. Any move from this position will offer some boxes to the opponent.

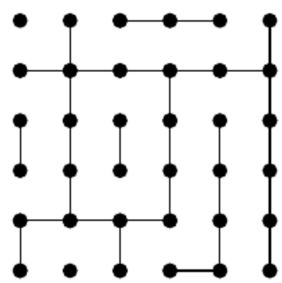

Figure 1: A Dots-and-Boxes endgame where any move will concede a box to the opponent.

Definition 3.1
a) A independent k-chain is a component consisting on a cycle of length k that includes vertex g.
b) An independent l-loop is a connected component consisting on a cycle of length l that does not include vertex g.
c) A simple endgame is a game where each component is either an independent chain or an independent loop.

The game shown in Figure 1 is a simple endgame: its only components are an independent 6-loop and several independent chains. We will only consider simple endgames in this part.

Then, we would consider keeping control in the Endgame, which is crucial in winning strategies. First, we define chains in Dots-and-Boxes games.

Definition 3.2 Chains of length 1 or 2 are called short chains, while chains of length $l \geq 3$ are called long chains, or simply chains.

Long chains are much more relevant in the game than short chains. This is the reason why when we write chains we will assume that we are referring exclusively to long chains.

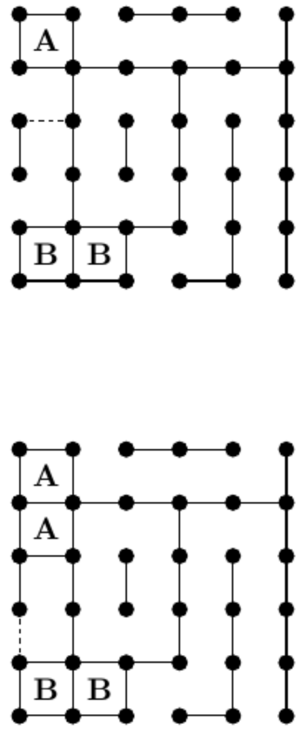

Figure 2: a) In this loony endgame, after taking the two boxes of the 2-chain at the bottom, Player B has to offer a (long) chain or a loop, and chooses to offer the 3-chain on the left. b) Player A, who is in control, only takes one box of the 3-chain.

Definition 3.3
a) A loony move is a move that offers at least 3 boxes. We will use the symbol to denote a loony option of a game.
b) A loony endgame is a game where all the available options are loony moves.
c) The player in control is the player who last played before reaching the loony endgame.
d) The strategy consisting in declining the last two boxes of each chain and the last four of each loop is called keeping control.

To understand **Definition 3.3**, let us see how the player in control plays the game on Figure 2:a) in order to minimize the number of boxes captured of his opponent. Note that it corresponds to the loony endgame obtained from the game in Figure 1 once the two short chains have been claimed. When a long chain is offered, the player in control will not take all the boxes but will decline the last two instead, as in Figure 2:b). In this way, he forces his opponent to play on another chain or loop. The player in control will play in an analogous way when offered a loop, but in this case he will decline 4 boxes, as in Figure 3.

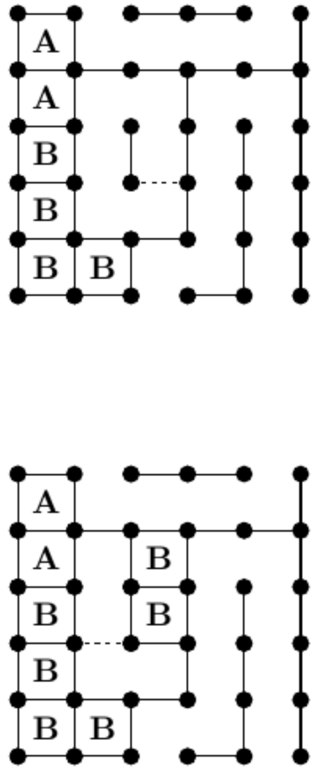

Figure 3: Player B offers a loop, then Player A declines the last four boxes.

Proposition 3.4 Let G be a loony endgame with n chains and m loops, for a total of b unclaimed boxes. Suppose that the net score (number of boxes claimed minus number of boxes claimed by opponent) for the player in control at this point is s. If $b + s > 4n + 8m$ then the player in control wins the game.

Proof. Keeping control guarantees the player in control all the boxes except 2n + 4m, that is, $b - 2n - 4m$ boxes, while his opponent captures 2n + 4m. Therefore, he obtains a net gain of $b - 2n - 4m - (2n + 4m) = b - 4n - 8m$ boxes in the loony endgame, which means that a sufficient condition for him to win is $b + s > 4n + 8m$. ■

Take the Figure 2:a) player Bob as an example. Bob is in control. Before any move, b=22, s=1, n=3, m=1, so b+s=21 and 4n+8m=20. Therefore, Bob wins by 1. In fact, in this example, Bob can win by more than 1 box because he does not need to decline any boxes in the last component he is offered.

We have just shown a strategy that will be a winning one in many games. When it is not, We show that one can probably win anyway, provided that one force his or her opponent to be the first to play in the loony endgame. That is because keeping control is not the only possible strategy.

Theorem 3.5 In a loony endgame G the player in control will, at least, claim half the remaining uncaptured boxes. Since this theorem we will use very often, we just call it **Half-Win**.

Proof. By definition of loony endgame, the first move in G must be a loony move, i.e., a move offering either a long chain or a loop.

Assume it is a k-chain. Let us call $G^{'}$ the rest of the game, i.e., G minus the k-chain. Let x be the net gain of the player who plays second in $G^{'}$. When offered the k-chain, we will consider two possible strategies for the player in control: either take all the boxes, or decline the last two. If he claims all boxes in the chain, he will play first in $G^{'}$, obtaining a total net gain of k-x (because his opponent will get a net gain of x in $G^{'}$, as he will play second there). If he declines the last two boxes in the chain, his opponent will play first in $G^{'}$, so the player in control will obtain a total net gain of k-4+x (because his net gain in the chain is k-4, since he claims k - 2 boxes and his opponent 2). If he chooses the strategy of those two that guarantees him a higher net gain, this net gain is $k-2+\mid x-2\mid$, as he will choose the first strategy if $x \leq 2$ and these condone if $x \geq 2$. Since k¿2,this net gain is positive, which means that he will claim more than half the uncaptured boxes.

The same applies when the player in control is offered an l-loop, only that in this case his decision depends on the sign of x-4, and his net gain will be $l-4+\mid x-4\mid$. This net gain is non-negative because $l \geq 4$, which implies that the player in control will claim, at least, half the remaining boxes. ■

3.2 Value of a Game

As we are considering loony endgames in this chapter, we will repeatedly refer to the player in control and his opponent.

Notation 3.6 *We call* Right *the player in control, and* Left *his opponent.*

When we have chains of length 3 in a loony endgame things tend to get more complicated. So, to simplify our study, it is convenient to distinguish between 3-chains and longer chains.

Definition 3.7 *A* very long chain*is a chain of length* ≥ 4*.*

To determine the best score differential that Right can obtain in a loony endgame, we consider the following definition.

Definition 3.8 *The value of a loony endgame* $G, V(G)$*, is the net gain for the player in control (that is, number of boxes claimed by him minus number of boxes claimed by his opponent) if he plays optimally.*

Two questions arise when faced with a loony endgame: which is the optimal play strategy for each player, and which is its value. Observe that, for Left, choosing an optimal move means choosing which string or loop to offer to his opponent. We already know that, if Right plays optimally, when offered a string (resp. loop) he will claim, at least, all the boxes but the last two (resp, four). Therefore, for Right, his only decision is if he must keep control by refusing those last boxes, or give up control by taking all the offered boxes. Thus, we can define the value of the game and the move as follows:

Definition 3.9 *Given a loony endgame* G *and a string or independent loop* H*, we write* $V(G|H)$ *to denote the net gain by Right assuming that we impose on Left to offer H in his next move. If, moreover, we assume that Right is enforced to keep control (and from then on both players play optimally), we write* $V_C(G|H)$ *to denote his net gain. On the other hand, if we assume that Right is enforced to take all boxes from H, thus giving up control, we will write* $V_G(G|H)$ *to denote his net gain.*

Since Left will offer the loop or string which minimizes the value of the game, we have that

$$V(G \mid H) = \min_{H\in\mathcal{H}} V(G \mid H)$$

where $\mathcal{H}$ is the set of all strings and loops in the loony endgame that Left can offer. On the other hand, as Right will keep control or not depending on which choice guarantees him a higher net gain,

$$V(G \mid H) = \max\{V_C(G \mid H), V_G(G \mid H)\}$$

3.3 Optimal Play by Left

When facing a loony endgame, Left must offer the independent chains in increasing order of length. Loops must also be offered in increasing order of length. Then, we will consider the optimal play by

Left.

First, we describe a powerful technique that is very useful to prove some results. The idea is simple: if you copy the moves of a perfect player, but with some exceptions, and obtain the same score as him, the moves you played are no worse that the ones played by the perfect player.

In order to understand man-in-the-middle, there is an example to illustrate it well. Assume that you are playing chess against two masters. In the first game, you use black pawns and in the second game you use white pawns. You apply the strategy of copying moves to another game in this way: Master 1, play white, and start in game 1. You use the same opening in round 2 and wait for the main round 2 to reply to that game. Then you copy this action on Game 1 and wait for Master 1's reply, and so on. In this way, you will draw with two masters or beat one of them. In fact, these two masters are playing their own games. You are just the middleman.

Using this idea, we can prove some results through the following process: We assume that a player (middleman) plays the same point and box game with two perfect players (called masters), but plays a different role in each game: In one game, he is the previous player, and in another game, he is the next player. He will not copy all moves like in the chess example, but replace some moves with moves that we claim are not worse than the substituted moves. If at the end of the game, the box claimed by the middleman in games where he did not replicate all the actions is no less than the masters in other games, we will prove our point.

Proposition 3.10 i) Given a loony endgame G containing two independent chains c and $c^{'}$, with respective lengths k ¡ $k^{'}$, it is never worse for Left to offer c than $c^{'}$, i.e., $V(G \mid c) \leq V(G \mid c^{'})$.

ii) Given a loony endgame G containing two independent loops l and l' of respective lengths k ¡ $k^{'}$, it is never worse for Left to offer l than $l^{'}$, i.e., $V(G \mid l) \leq$ V (G | $l^{'}$).

Proof. We prove the result for independent chains. The proof for loops is analogous.

Assume the man-in-the-middle is playing Go against two sensei. He copies all the moves of the sensei, except in the following situation: if Sensei A offers the longer chain c in Game 1 when no sensei has still offered c in his game, then the man-in-the-middle offers the shorter chain c in Game 2. Sensei B will either keep control by declining the last 2 boxes, or claim all of them. The man-in-the-middle does likewise in Game 1. At this point, the man-in-the-middle has claimed k k more boxes in Game 1 than Sensei B in Game 2. The man-in-the-middle continues copying moves.

At some point before the end of the game, one of these two things will happen first:

i) Sensei A offers c in Game 1. Then the man-in-the middle offers c in Game 2, waits for the reply of Sensei B, and follows the same strategy (keep control or claim all) in Game 1. As a result, the man-in-the middle has the same score in Game 2 as Sensei A in Game 1.

ii) Sensei B offers c in Game 2. Then the man-in-the middle offers c in Game 1, waits for the reply of Sensei B, and follows the same strategy (keep control or claim all) in Game 1. As a result, the man-in-the middle has the same score in Game 2 as Sensei A in Game 1.

In either case, from that point on until the end of the game, the man-in-the-middle copies the moves of the corresponding sensei. As a result, the number of boxes claimed by the man-in-the-middle in Game 1 is no less than the number of boxes claimed by Sensei B in Game 2, which proves that his initial deviation from the copy strategy (offering c) is no worse than an optimal move (the offering of c by Sensei A). ■

In this case, we need to notice the length of chains. Considering for the winning strategies of Left, we offer a shorter chain as priority, which will lead to strictly a better result.

Then, we consider the situations in loony endgame by Left. Another known result that applies to arbitrary loony endgames is that it is almost always preferable (better or equal) for Left to offer a loop than a longer chain.

Proposition 3.11 In any loony endgame G containing an independent k-chain c and an independent $k^{'}$-loop l, $k^{'}$ $\leq$ k and k $>$ 4,$V(G \mid l) \leq V(G \mid c)$.

We can also improve Proposition for simple loony endgames, since it is always better for Left to offer any loop than any very long chain.

Proposition 3.12 If G is a simple loony endgame containing a loop l and a very long chain c, then $V(G \mid l) \leq V(G \mid c)$.

The proof of **Proposition 3.11** and **Proposition 3.12** is very similar to the proof of man-in-the-middle and **Proposition 3.10**.

Thus, we can conclude that:

i) If G contains no loops, an optimal move for Left is to offer the shortest chain c , i.e., $V(G) = V(G \mid c)$.
ii) If G contains loops but no 3-chains, an optimal move for Left is to offer the shortest loop l , i.e., $V(G) = V(G \mid l)$.
iii) If G contains both loops and 3-chains, either offering a 3-chain c or offering the shortest loop l is an optimal move for Left, i.e., $V(G) = \min\{V(G \mid c), V(G \mid l)\}$.

3.4 Optimal Play by Right

Now let us face the question of what must be the answer of Right when offered a string or loop. As we already observed, his only decision when offered a string is if he must decline the last two boxes in order to keep control, or if it is better for him to claim them and give up control. Analogously, when offered a loop he only has to choose between declining or not the last four boxes.

As As we showed in **Half-Win**, the decision of keeping control or claiming all boxes depends on the value of the rest of the game.
Proposition 3.14 *Given a loony endgame G, assume Left offers a string or a loop H. Let $G^{'}$ be the rest of the game, $G = H + G^{'}$*

i) If H is a string, Right's optimal strategy is to keep control if the value of the rest of the game is at least 2, and to claim all the boxes of the chain otherwise, i.e.,

$$V(G \mid H) = V_C(G \mid H) \Leftrightarrow V(G^{'}) \geq 2$$

ii) If H is a loop, Right's optimal strategy is to keep control if the value of the rest of the game is at least 4, and to claim all the boxes of the loop otherwise, i.e.,

$$V(G \mid H) = V_C(G \mid H) \Leftrightarrow V(G^{'}) \geq 4$$

Proof. If H is a string of length l, keeping control means a net gain in the string of $s-4$ boxes ($s-2$ claimed by Right, minus 2 claimed by Left) and being in control in the remaining of the game, $G^{'}$. That gives Right a total net gain of $V_C(G \mid H) = s - 4 + V(G^{'})$, while claiming all s boxes and giving up control to his opponent gives a net gain of $V_G(G \mid H) = s - V(G^{'})$. Therefore it is optimal for Right to keep control when $c - 4 + V(G^{'})s - V(G^{'})$, that is, when $V(G^{'}) \geq 2$.

On the other hand, if H is a loop of length l, keeping control gives Right a net gain of $V_C(G \mid H) = l - 8 + V(G^{'})$, while claiming all the boxes and giving up control to his opponent gives him a net gain of $V_G(G \mid H) = l - V(G^{'})$. Therefore it is optimal for Right to keep control when $V(G^{'}) \geq 4$.■

Note that when H is a string and $V(G^{'}) = 2$, as well as when H is a loop and $V(G^{'}) = 4$, it is indifferent to keep or give up control. We can simply assume that Right keeps control in this case.

4 Conclusion

In this paper, we discuss the basic definitions of scoring games and well-tempered scoring games and describe specifically the rules and winning strategies of Dots and Boxes. However, the winning strategies of well-tempered games, especially for i-games, haven't been discussed. It is meaningful to research one day. Since some of proof is not rigorous, the reader can point it out and send it to us, making it possible for us to improve the proof.

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