GAMES WITH ENTAILING MOVES

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Abstract. In this paper, we discuss the theory behind games with entailing moves, defining new positions and introducing different tactics. We will later apply this knowledge to the ruleset Top Entails.

1. Introduction

In most combinatorial games, certain axioms that are defined: players alternate after each move, whoever cannot move loses, there are no draws, and games can be disjunctively added together. However, what happens if go against the axioms of alternating play and disjunctive sums?

We shall include entailing moves in our classical games. Entailing moves force a move out of a player, which, in turn, limits the number of responses for a player. Examples of entailing moves are the following: a player must play again or a player must play in the same component as the previous player did.

Note that the values of individual components in a game with entailing moves will not be relevant, since we cannot disjunctively add games. We will discuss how to extend our definitions in classical games in the next few sections.

2. Affine Normal Play

Let us define the values of games with entailing moves using infinities. Games will be recursively constructed in a different way than classical games. Denote *affine* normal play forms as $\mathbb{NP}^\infty$. Games are recursively created from the game $\{\infty \mid \overline{\infty}\} = 0$.

As with classical normal play, the outcome classes are $\mathcal{L}$, $\mathcal{R}$, $\mathcal{N}$, and $\mathcal{P}$. From Left's perspective, the best outcome class is $\mathcal{L}$, while the worst is $\mathcal{R}$. $\mathcal{N}$ and $\mathcal{P}$ are indistinguishable.

There are, however, some key differences that we shall define with the following axioms:

Axiom 1. The infinities satisfy:

1) $\infty \in \mathcal{L}$.
2) $\overline{\infty} \in \mathcal{R}$.
3) For all $X \in \mathbb{NP}^\infty \setminus \{\overline{\infty}\}$, $\infty + X = \infty$.
4) For all $X \in \mathbb{NP}^\infty \setminus \{\infty\}$, $\overline{\infty} + X = \overline{\infty}$.
5) $\infty + \overline{\infty}$ is not defined.

We can define the comparison between games similarly to games in classical normal play.

Definition 2.1. Let $G, H \in \mathbb{NP}^\infty$. $G \geq H$ if for every game $X \in \mathbb{NP}^\infty \setminus \{\infty, \overline{\infty}\}$, $o(G+X) \geq o(H+X)$. Furthermore, $G = H$ if $G \geq H$ and $H \geq G$.

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We must exclude the infinities to prevent cases such as $o(\infty + \overline{\infty})$, which is undefined according to Axiom 5.

From the previous axioms and Definition 2.1, we can formulate an important result.

Theorem 2.2. *Let $G \in \mathbb{NP}^{\infty} \setminus \{\infty, \overline{\infty}\}$. Then, $\infty \geq G$ and $G \geq \overline{\infty}$.*

Proof. To show that $\infty \geq G$, we must show that $o(\infty + X) \geq o(G + X)$ for any $X \in \mathbb{NP}^{\infty} \setminus \{\infty, \overline{\infty}\}$. From Axioms 1 and 3, we know that $o(\infty + X) = o(\infty) \in \mathcal{L}$. Thus, $\infty \geq G$.

To show that $G \geq \overline{\infty}$, we must show that $o(G+X) \geq o(\overline{\infty}+X)$ for any $X \in \mathbb{NP}^{\infty} \setminus \{\infty, \overline{\infty}\}$. From Axioms 2 and 4, we know that $o(\overline{\infty} + X) = o(\overline{\infty}) \in \mathcal{R}$. Since every affine normal form is greater than or equal to an $\mathcal{R}$ position, $G \geq \overline{\infty}$. ■

Definition 2.3. (Quiet Game) Let $G \in \mathbb{NP}^{\infty}$. If $G \notin \{\infty, \overline{\infty}\}$, $\infty \notin G^L$, and $\overline{\infty} \notin G^R$, then G is *quiet*.

See how no player can move to a winning subposition in a *quiet* game, since none of the options and the game itself are infinities.

We will now introduce one of the most important theorems in affine normal play.

Theorem 2.4. *(Fundamental Theorem of Affine Normal Play) If $G \in \mathbb{NP}^{\infty}$, then $G \geq 0$ iff $G \in \mathcal{L} \cup \mathcal{P}$.*

Proof. We must show that it is sufficient for $G \in \mathcal{L} \cup \mathcal{P}$ for $G \geq 0$. This is true, based on the fact that $0 \in \mathcal{P}$ and $\mathcal{L}$ position is always greater than an $\mathcal{P}$ position from the order of outcomes.

Now we must show that it is necessary for $G \in \mathcal{L} \cup \mathcal{P}$ for $G \geq 0$. Assume that $G \in \mathcal{L} \cup \mathcal{P}$. By Theorem 3, if $G = \infty$, then $G \geq 0$. Finally, we must show that the condition holds when $G \neq \infty$ and $G \in \mathcal{L} \cup \mathcal{P}$.

Let $X \in \mathbb{NP}^{\infty} \setminus \{\infty, \overline{\infty}\}$. Let Left start first in X. If the winning move is to X^L, then Left can mimic that same move in $G + X$. Since the assumption $G \in \mathcal{L} \cup \mathcal{P}$ holds, Left wins going first in $G + X$. Assume Left wins when going second in X. When going second in $G + X$, Left can just play in the same component as Right with winning moves, again from the assumption that $G \in \mathcal{L} \cup \mathcal{P}$. Thus, Left wins going second in $G + X$.
Since $o(G + X) \geq o(X)$, $G \geq 0$. ■

We will also define the conjugate of a game in $\mathbb{NP}^{\infty}$, which is similar to the negation of a game in $\mathbb{NP}$ (the set of classical normal play forms) but with a few modifications.

Definition 2.5. (Conjugate) The conjugate of $G \in \mathbb{NP}^{\infty}$ is

$$conj(G) = \begin{cases} \overline{\infty} \text{ if } G = \infty \\ \infty \text{ if } G = \overline{\infty} \\ Otherwise, \{G^R \mid G^L\} \end{cases}$$

3. Affine Impartial Theory

Now that we have a basic understanding of how affine normal play works, we can delve into a specific set of affine normal play games, namely affine impartial games. We will extend the Sprague-Grundy theory and *mex* rule from classical impartial theory. We will also look into new positions that arise from affine impartial games, namely sunny and loony positions.

Definition 3.1. (Symmetric Game) Consider a game $G \in \mathbb{NP}^{\infty}$. Then, G is *symmetric* if $G \notin \{\infty, \overline{\infty}\}$ and $G^R = conj(G^L)$.

Note that G cannot be equal to either of the infinities, since these infinities do not have birthdays are on day 0 or later.

We can now define an affine imparial game, which is the main focus of this section.

Definition 3.2. (Affine Impartial) A game $G \in \mathbb{NP}^{\infty}$ is affine impartial if it is symmetric and all its quiet options are symmetric as well. We define this subset of affine impartial games as $\mathbb{IM}^{\infty} \subseteq \mathbb{NP}^{\infty}$.

G must be symmetric, since Left and Right must have the same options to play. All the quiet subpositions must also be symmetric, since the game has not ended yet. If a non-quiet game is played, then the outcome of the game is already determined.

Remark 3.3. Let $\mathbb{NIM} \subseteq \mathbb{IM}^{\infty}$ This subset contains all the positions in $\mathbb{IM}^{\infty}$ that are equal to nimbers.

We shall also define the equality between games that are part of the affine impartial subset. We will do this in terms of modulo $\mathbb{IM}^{\infty}$.

Definition 3.4. Let $G, H \in \mathbb{IM}^{\infty}$. $G =_{\mathbb{IM}^{\infty}} H$ if, for every form $X \in \mathbb{IM}^{\infty}, o(G + X) = o(H + X)$.

Notice how this is definition is almost analogous to the other definitions of equality between 2 games.

We can now look into sunny and loony positions, which result from allowing the use of entailing moves in impartial games. This is shown in [1]

Definition 3.5. Let a sunny position $\odot$ be $*0$ from the set of all nimbers. Let a loony position ☽ be an empty set with no nimbers.

It is easy to understand that sunny positions are winning positions for the player who moves to this position, while loony positions automatically are losing positions for the player to move to this position. Thus, $\odot \in \mathcal{P}$ and ☽ $\in \mathcal{N}$.

Due to the properties of loony positions, we can define a few axioms:

Axiom 2. (Properties of Loony Positions)

1) ☽ $+ *n =$ ☽ for $n \in \mathbb{Z}^+$.
2) ☽ $+$ ☽ $=$ ☽.
3) ☽ $= \{\infty \mid \overline{\infty}\}$.
4) $\mathcal{G}($☽$) = \infty$.

Notice how ☽ absorbs any nimber and itself. It has similar properties to ☆, as both are idempotents.

Before we look into the extension of the Sprague-Grundy Theory, we must define 2 types of nimbers.

Definition 3.6. (Immediate Nimbers) Let $G \in \mathbb{IM}^{\infty}$. The set of *G-immediate nimbers* is $S_G = G^L \cap \mathbb{NIM} = G^R \cap \mathbb{NIM}$.

Before we can define the other type of nimber, we must define what Left and Right checks are.

Definition 3.7. (Check Games) Let $G \in \mathbb{NP}^{\infty}$. If $\infty \in G^L$, then G is a *Left-check*. If $\overline{\infty} \in G^R$, then G is a *Right-check*. If G is either a *Left-check* or *Right-check*, then G is a *check*. Let $G^{\overrightarrow{L}}$ be a Left option of G that is Left-check, while $G^{\overleftarrow{R}}$ is a Right option of G that is a Right-check.

Definition 3.8. (Protected Nimbers) Consider a game $G \in \mathbb{IM}^{\infty}$. The set of *G-protected nimbers* P_G is:
1) $P_G = \mathbb{NIM}$ if $\infty \in G^L$.
2) $P_G = \{*n : G^L + *n \in \mathcal{L}, G^{\overrightarrow{L}} \in G^L\}$, otherwise.

In simpler words, the set of G-protected nimbers is the set of nimbers in affine impartial play if G is a Left-check. If G is not a Left-check, then the set of G-protected nimbers is the nimbers such that G^L plus the nimber is an $\mathcal{L}$ position and a Left-check is part of the Left options of G.

We can now look into the main theorem of this paper, namely the extension of the Spargue-Grundy Theory, or the Affine Impartial Minimum Excluded Rule.

Theorem 3.9. *(Affine Impartial Minimum Excluded Rule) Let* $G \in \mathbb{IM}^{\infty}$. *We have the following possibilities:*
1) If $S_G \cup P_G = \mathbb{NIM}$*, then* $G = \rightmoon$ *and* $mex(\mathcal{G}(S_G \cup P_G)) = \infty$.
2) If $S_G \cup P_G \neq \mathbb{NIM}$*, then* $G = *(mex(\mathcal{G}(S_G \cup P_G)))$.

Proof. Check the proof from page 14 in [2]. ∎

4. Application - Top Entails

We will now apply all the theory we have discussed previously to a ruleset with entailing moves called Top Entails. This is played with piles of tokens. It follows the rule of alternating play. On a move, a player can remove the top token from any pile or split a pile into 2 nonempty piles. If the player chooses the former move, the next player must make a move in the same heap. Let us start with a few examples to get used to the ruleset.

Let us say we start off with no heaps of tokens. If Left starts, she has no move, so the Left option is $\overline{\infty}$, since Right wins. The case when Right starts first is symmetric to the previous case, so the Right option is ∞. Thus, our game is $\{\overline{\infty} \mid \infty\} = 0$. It also turns out that this game is $\odot$, as this is $*0$.

We will now consider the case with a heap of 1 token. This is the best possible position for a player who starts first. The player who goes first can take the 1 token, forcing the next player to play in the same component and win. Thus, this game will be $\{\infty \mid \overline{\infty}\} = \rightmoon$.

Things get a little complex with a heap with 2 tokens. The heap can be split up into 2 piles of 1 token each or one token can be taken from the top of the current heap. In the former case, the next player can play in either of the 2 heaps with 1 token each. Either move produces a similar outcome of playing in a heap of 1 token, so this move goes to a $\rightmoon$ game. In the latter case, the value will also be $\rightmoon$, since the game reduces down to $\{\infty \mid \overline{\infty}\}$. Thus, one pile of 2 tokens can be represented as $\{\rightmoon \mid \rightmoon\}$, which is a $\odot$ position.

There are more positions that can be calculated using a similar method as above. This can be found in [2] and/or [1].

5. Other Applications

Top Entails is one of the easier rulesets to become acquainted with. However, there are other rulesets with similar rules that can be analyzed Most of them are found in [1].

(1) Nimstring: This game involves a finite set of lattice points. On a player's turn, they must connect 2 horizontally or vertically adjacent points that do not already have a connection. If a 1 x 1 square is completed, the same player who made the square must play again. If they cannot, they lose. This goes against the classical rule of alternating play, since the a player can play immediately after making the square.
(2) Goldbach's Nim: This is a little more complicated than the other 2 we mentioned. The setup is the same as Nim. If a player takes a prime number of items from any pile or leaves a prime number of items in any pile, then the next player must play in that same pile. This game utilizes *Goldbach's Conjecture*, which states that any even number can be written as the sum of two primes.
(3) Dots and Boxes: This has similar rules to Nimstring. It is played on a finite grid. On a player's turn, they must connect two adjacent points that do not already have a line between them. The player who completes the 4th side of a 1 x 1 box gets a point and another turn. After no moves are available, the game ends. The player with the most points wins. Not only does this game go against the classical rule of alternating play, but it also goes against how a classical game is won, since classical combinatorial games are won based on which player makes the last move.

References

[1] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Chapter 12: Games Eternal - Games Entailed*, volume 2. A.K. Peters, 2004.
[2] Urban Larsson, Richard J Nowakowski, and Carlos P Santos. Impartial games with entailing moves. *arXiv preprint arXiv:2101.11699*, 2021.