Cops and Robbers

Varun Rao, Edward Beckon, and Simon Seignourel

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ABSTRACT. Cops and robbers is a graph-pursuit game between two players: one controlling a robber and the other controlling one or more cops. We examine the cop number $c(G)$ of various graphs and explore existing bounds on it. In addition, we provide a few results of our own and discuss a variation to the game.

1 Introduction

The game of cops and robbers in played on a graph $G = (V, E)$, between two players: the cops (addressed as *she*) and the robber (addressed as *he*). At the beginning, the cops place themselves at vertices $v \in V$, and the robber places himself on one as well. These positions are known to both players. Then, starting with the cops, the cops and robber make successive moves, alternating with each other. Each move by the cops consists of moving a cop from one vertex to another along an edge in the graph, or staying still. Similarly, each move by the robber moves him from one vertex to another adjacent one or lets him stay still.

The game ends when a cop "catches" the robber, or when she occupies the same vertex as him. The goal of the cops is to make moves to catch the robber, and the goal of the robber is to evade capture indefinitely.

2 Terminology

In cops and robbers, we have the following outcome classes:

Definition 2.1. A graph G is a *k-cop-win* if there exists a set $C = (v_1, v_2, \ldots v_k)$ of vertices in G such that placing k cops at the vertices in C guarantees that no matter where the robber lands, he will eventually be caught.

Definition 2.2. A graph G is a *k-robber-win* if it is not a k-cop-win. In other words, for any placement of k cops, there is an infinite sequence of steps that the robber can follow to evade capture.

1-cop-wins and 1-robber-wins are simply referred to as cop-wins and robber-wins. Note that if G is a k-cop-win, it is a $(k+1)$-cop-win because we can just add an extra cop to some vertex and have it not move at all. Conversely, if G is a k-robber-win, it is a $(k-1)$-robber-win because it becomes increasingly difficult to capture the robber with fewer cops. Therefore, we can construct the following definition:

Definition 2.3. The *cop number* $c(G)$ of a graph G is the minimum number of cops needed to guarantee that the robber will eventually be caught, no matter where he lands. For any integer k, G is a k-robber-win if $k < c(G)$ and G is a k-cop-win if $k \geq c(G)$.

The cop number is one of the largest areas of research in cops and robbers.

3 Cop Numbers of Simple Graphs

Definition 3.1. A graph $G = (V, E)$ is *connected* if for any $u, v \in V$, there is a series of edges that can be followed to go from u to v.

Definition 3.2. A graph $G = (V, E)$ is *disconnected* if it is not connected. The *connected components* $C_1, C_2, \ldots, C_k$ of G are the connected subsets of G satisfying $V_{C_1} \cup V_{C_2} \cup \cdots \cup V_{C_k} = V$ and no $v \in V$ is in more than one C_i.

Using these definitions, we have

Proposition 3.3. *If G is a disconnected graph and $C_1, C_2, \ldots, C_k$ are its connected components, $c(G) = c(C_1) + c(C_2) + \cdots + c(C_k)$.*

Since neither the cops nor the robber can travel from one connected component to another, we need $c(C_i)$ cops in each component to guarantee that the robber is caught.

Proposition 3.3 tells us that we are only interested finding the cop numbers of connected graphs, since disconnected graphs can be calculated in terms of their connected components. Having established this result, we can start to find the cop numbers of certain classes of graphs.

Definition 3.4. A *finite path* is a graph $G = (V, E)$ where if $V = (v_1, v_2, \ldots, v_n)$, we have $E = \{(v_i, v_{i+1}) \mid 1 \leq i < n\}$. In other words, it is a finite set of vertices connected in a straight line.

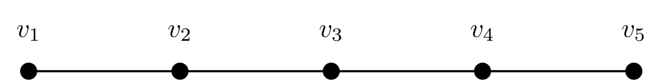


Figure 3.1: A finite path with 5 vertices.

We claim that all finite paths have $c(G) = 1$. This is because no matter where the cop chooses to start, there will be finitely many vertices to the left and right of her. The cop can just keep making moves towards the robber, and she will eventually catch him because he has nowhere to run.

Definition 3.5. A *cycle* is a path with an extra edge between the two endpoints. To construct a cycle of length 5 from Figure 1.1, we can draw an edge from v_5 to v_1.

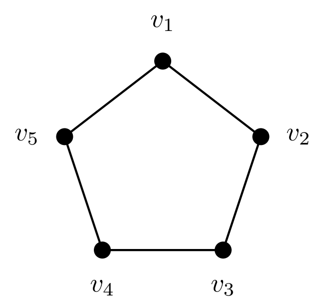


Figure 3.2: A cycle of length 5.

If G is a cycle of length greater than 3, then $c(G) = 2$. With one cop, the robber has a winning strategy. He chooses the vertex opposite the cop, or one of the two vertices opposite her if the number of vertices is odd. Because the cycle has length greater than 3, the cop needs at least two moves to reach the robber's current position. When the cop moves, the robber moves in the same direction along the cycle, away from the cop. If the cop ever chooses to not move, the robber does likewise. The robber can follow this strategy for infinitely many turns, so $c(G) > 1$.

However, two cops are enough: after the robber chooses their vertex, the cop to the robber's right moves clockwise and the cop to the robber's left moves counterclockwise. The robber is trapped and will eventually be caught. Therefore, $c(G) = 2$.

It turns out we can generalize the statement about paths to a larger class of graphs: trees.

Definition 3.6. A tree is a graph $G = (V, E)$ such for every $u, v \in V$, there is exactly one path from u to v. Every pair of vertices is connected, but no subset of vertices and the edges between them forms a cycle.

Proposition 3.7. *If $G = (V, E)$ is a tree, $c(G) = 1$.*

Proof. Suppose that a single cop is placed at v_c and the robber at v_r, where $v_c, v_r \in V$. Consider the set of vertices $R \subset V$ such that the path from any vertex in R to v_r does not include v_c. In other words, the robber can safely reach any $v \in R$ without getting caught, assuming that the cop does not move. Note that any move that the robber makes cannot change R.

We will show that the cop always has a move from v_c to some v_c' which reduces R to some $R' \subset R$, therefore guaranteeing that the robber will eventually be caught. To do this, consider the path from v_c to v_r, and define v_c' as the vertex next along the path from v_c. The robber still cannot reach any vertex in

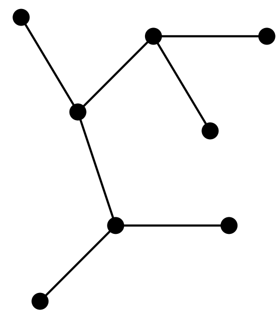


Figure 3.3: A sample tree.

$V \setminus R$ because he will have to go through v_c, and therefore v'_c. We have shown that $R' \subseteq R$.

Next, $v'_c \in R$ but $v'_c \notin R'$, so we have $R' \subseteq R \setminus v'_c$. Thus, $R' \subset R$, so the cop always has a move to reduce R. □

4 Bounds on the cop number

It is more difficult to calculate the cop numbers of larger classes of graphs, so bounds have been provided instead, including generalizations about all graphs.

Some are instantly clear: for example, the cop number of any graph is less than the number of its vertices. This is because if a cop is placed at every vertex, then the robber cannot possibly escape. Furthermore, if there is always a cop neighboring the robber when the robber is first placed, then the robber cannot escape. We can clarify this with a definition:

Definition 4.1. For any graph $G = (V, E)$, $\gamma(G)$ is the smallest possible set of vertices such that for all $v \in V$ there is some $u \in N(v)$ ($u = v$ is possible) such that $u \in \gamma(G)$.

Proposition 4.2. *$c(G) \leq \gamma(G)$.*

In other words, $\gamma(G)$ is the smallest set of vertices that will always be neighboring every open vertex. This means that no matter where the robber places himself on his first turn, a cop can capture him on the next. The cop number must of course be less than or equal to $\gamma(G)$, but we want to find more specific ways to bound the cop number.

One thing we can do is attempt to construct a bound on the cop number of a graph in terms of the sum of the cop numbers of is subgraphs. This allows us to perform something analagous to a connected sum of two graphs. We have

Theorem 4.3. *For any graph $G = (V_G, E_G)$ and partition $C = (V_S, V_T)$ of V_G, define*

$$S = V_S, \{(u, v) \in E \mid u, v \in V_S\},$$

$$T = V_T, \{(u, v) \in E \mid u, v \in V_T\},$$
$$E_C = \{(u, v) \in E \mid u \in V_S, v \in V_T\}.$$

Then, we have

$$c(G) \leq c(S) + c(T) + |E_C|.$$

Proof. Suppose we are playing as the cops, and we have a strategy to win in S (with $c(S)$ cops) and one to win in T (with $c(T)$ cops). Then, we can construct a strategy to win in G with $c(S) + c(T) + |E_C|$ cops. To start, place $c(S)$ cops in S according to the strategy for S, and do the same for T. Then, for every $(u, v) \in E_C$, place one of the remaining $|E_C|$ cops at u.

We'll assume without loss of generality that the robber chooses a vertex in S to start. (If he doesn't, we can simply swap S and T.) While the robber doesn't move to T, we can follow the strategy for S, guaranteeing a win if he stays in S. If the robber attempts to move from S to T along an edge $(u, v) \in E_C$, the game will end because there is a cop at u that has not moved from the beginning of the game. The robber either tries to move from u to v, which is impossible since the game will end once before he even moves to v, or from v to u, which will also end the game once he is at u. Thus, if the robber lands in S, he cannot move to T, but he will also lose by staying in S since there are $c(S)$ cops there. □

One of the best bounds on a large class of graphs was shown by Aigner and Fromme. To state it, we first need a definition, as well as a useful lemma:

Definition 4.4. A graph G is *planar* if it can be drawn on the 2D plane without intersecting edges.

Lemma 4.5. *Let $G = (V_G, E_G)$ be any graph, u,$v \in V_G$, $u \neq v$ and $P = \{u, v_1, ..., v_t = v\}$ a shortest path between u and v. Then a single cop C on P can, after a finite number of moves, prevent the robber R from entering P. That is, R will be immediately caught if he moves onto P.*

Proof. We call the shortest path between u and v $P = \{u = 0, 1, 2, ..., t = v\}$, and say that $d(u, v)$ is length of the shortest path between vertices u and v. We also say that after the cop moves, she is on vertex $c \in V(P)$ and the robber is on $r \in V(G)$ (or any vertex in the game). Furthermore, we'll assume that for any vertex z on $V(P)$ (on the protected path):

$$d(r, z) \geq d(c, z) \text{ for all } z \in V(P).$$

In other words, we're assuming that the cop is closer than the robber to any vertex z on $V(P)$. We then claim that the cop can always preserve this condition of being closer to z.

If the robber stays put, then so does the cop. If the robber moves from a position r to s, then

$$d(s, z) \geq d(r, z) - 1 \geq d(c, z) - 1 \text{ for all } z \in V(P).$$

This means that the distance from the robber's new position s to our goal z cannot be more than 1 edge closer than r was. Since $d(r, z) \geq d(c, z)$, then we know the cop is only one edge away from preserving the condition, and she can do so on her turn.

The robber would only be threatening if he was closer than the cop to two vertices on either side of her. This would mean the robber could pass over the cop at any point and the path would not be guarded. However, this is impossible because P is the shortest path between u and v. Therefore, there would be no alternate route the robber could pass by that is closer to two vertices on either side of the cop at once.

Finally, we want to show that if the cop is further from z than the robber, then after a finite number of moves this condition can be reached (rendering the path impossible to cross). This is not difficult: the cop can simply move closer to z every turn. This way, at some point the cop will be equally far from the robber and can just continue following the robber's moves to block off the path. □

We've now shown that in a finite number of moves, a cop can "guard" a certain path between two vertices such that the robber will be prevented from entering it. This will be useful in the proof to come, because it helps bound the area the robber can move to on a certain graph.

We now present the result of Aigner and Fromme: (See **[AF84]**)

Theorem 4.6. *If G is a planar graph, $c(G) \leq 3$.*

Proof. We now want to prove that for any finite planar graph, $c(G) \leq 3$. To do so, we're going to prove that for any planar graph with 3 cops, we can always reduce the territory of the robber incrementally. Each stage of the game i has for the robber R a subgraph R_i, which is the *robber territory* or all vertices which the robber may still safely enter. We therefore want to show that after a finite number of moves R_i is reduced to $R_{i+1} \subsetneq R_i$. Thus, R_{i+1} is strictly smaller than R_i.

Let's suppose that after the robber's move we can have two possible situations:

1. We have a cop C on vertex u where u is the only vertex connecting R_i to the rest of the graph. This means that R_i is the component of $G - u$ containing the robber's vertex r.

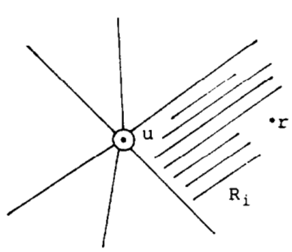


Fig. 8.

[AF84]

2. There are two vertices u and v with two paths P_1 and P_2 connecting them. These paths are disjoint, and partition G into an exterior and an interior. The robber occupies a vertex r in the exterior region E. We also assume that P_1 and P_2 are the first and second shortest u, v paths in $P_1 \cup P_2 \cup E$. One Cop C_1, placed on a vertex $c_1 \in V(P_1)$, controls P_1 by the lemma, and a cop C_2 on $c_2 \in V(P_2)$ controls P_2 the same way. The robber territory $R_i = E$. Case b can only truly exist in a planar graph where the inner and outer portion are only accessible by passing through P_1 or P_2. Otherwise they are not adequately guarded by the cops and the proof breaks down.

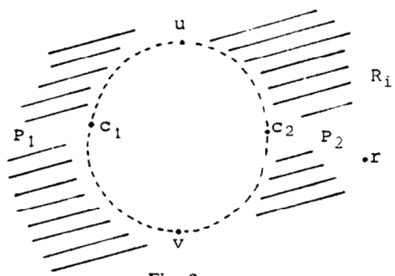


Fig. 9.

[AF84]

Let's begin with case a. If u only has one neighbor v in R_i (one adjacent vertex), then u can move to v. Thus, the robber territory is reduced and we return to case a with $R_{i+1} \subsetneqq R_i$. Suppose, then, that u has at least two neighbors a and b in R_i, and let P be the shortest path between a and b. As long as the cop C on vertex u stays there containing the robber in R_i, then another free cop can always move over and control P after a finite number of moves. This is true by the lemma. We are now at case b: we have a path $P_1 = a, u, b$ controlled by the first cop and a path $P_2 = P$ controlled by a second cop. Again, the robber territory is reduced: since $R_{i+1} \subseteq R_i - V(P)$ (meaning that R_{i+1} is a subset of and may equal $R_i - V(P)$), then it must be true that $R_{i+1} \subsetneqq R_i$.

We must now examine case b. First, let's suppose there is no path in the exterior portion R_i connecting u to v other than P_1 and P_2. This means that R_i is simply a bunch of disjoint components attached to the vertices of P_1 and P_2. The robber's vertex r must be in one of these disjoint components, attached to either P_1 or P_2 by a vertex a. If cops C_1 and C_2 continue guarding P_1 and P_2, then the third cop C_3 can guard a. We now return to case a where vertex $u = a$ and in this way $R_{i+1} \subsetneqq R_i$.

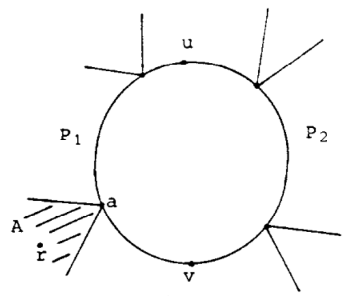


Fig. 10.

[AF84]

We should now examine the case where P_1 and P_2 are not the only paths connecting u and v. Call P_3 another u, v path in $R_i \cup P_1 \cup P_2$. P_3 could jut out from P_1 or P_2, be entirely disjoint from them, or connect to any one of the vertices on P_1 and P_2.

No matter where P_3 is, C_1 and C_2 can keep guarding paths P_1 and P_2 by the lemma while the third cop C_3 can move over to P_3. Also by the lemma, the third cop can thus control P_3. The robber at this point could be outside P_3 or between P_3 and P_1 or P_2: either way, the robber's territory is reduced and $R_{i+1} \subsetneqq R_i$. It is also important here that the graph is planar, otherwise the inner portion could be accessible to the robber without crossing through P_3. This means the robber territory would not necessarily be reduced.

Overall, this proof shows that in all possible cases, three cops can manipulate their positions so that they reduce the robber territory in a finite number of turns. If the robber territory can always be reduced with three cops, then eventually the robber will have nowhere to go by induction and the cops win. Therefore, $c(G) \leq 3$ for any planar graph. □

5 Variations to the game

There are several interesting variations to the game of cops and robbers. One of these is Drunken Robber, where the robber chooses a random edge to move along.

As one may immediately speculate, this robber is much easier to catch. In fact, this robber can always be caught. This is why in drunken cops and robbers, the cop number is not interesting and we instead consider the minimum time needed to catch the robber. We have

Definition 5.1. $ct(G, k)$ is the capture time for a graph G in regular cops and robbers, where there are k cops.

Definition 5.2. $dct(G, k)$ is the expected capture time in a game G of drunken cops and robbers where there are k cops.

Theorem 5.3. $dct(G, k) < \infty$ for any connected graph G and $k \geq 1$

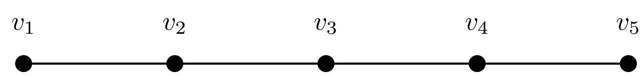
The proof for this theorem involves a good amount of probability theory. At its core, it proves that even if a single cop stayed on one vertex, the robber would eventually reach that vertex because it is moving randomly. Although this is a sub-optimal strategy for the cops, it shows that the robber will always be caught.

For example, a cycle graph only needs one cop with a drunken robber. Since the robber is moving randomly, he will not always move in one direction like a logical robber would: at some point he will stay still or move backwards, thus lowering the space between him and the cop. Since the cop is moving forward once per turn, she is guaranteed to catch him in a much shorter amount of time. With two cops, this is also evident: it takes a much shorter amount of time to catch the robber. The ratio of the capture time for a logical versus a drunken robber is defined as the cost of drunkenness, or the robber's loss of time when drunk.

Definition 5.4. We define the *cost of drunkenness* for a graph G as $\frac{ct(G,k)}{dct(G,k)}$.

Usually, the cost of drunkenness is this value with $c(G)$ cops. This means for a cycle with four or more vertices the cost of drunkenness is for a graph with two cops. With two cops, $ct(G)$ is about one fourth of the amount of vertices. However, for a drunken robber this value is moderately reduced, as this robber will not necessarily stay put the entire time.

Example. We can calculate $ct(G, c(G))$, $dct(G, c(G))$, and the cost of drunkenness for a path with 5 vertices.



The cop will always start by placing herself in the center. If the robber is sober, he will go to an edge vertex and remain there. This makes the capture time 3: one turn for the cop to move, then the robber stays still, then the cop moves onto his vertex.

If the robber is drunk, he could place himself on any of the five vertices. If he places himself in the middle, the capture time is 0. If he places himself adjacent to the cop, the capture time is 1. If he goes to an edge, then after the cop moves toward him there's equal chance he will stay still or move toward the cop. This means the capture time will either be 2 or 3: if we find the average of these two, we get a capture time of 2.5 for both edge vertices. All in all, we find that

$$dct(G,1) = \frac{2.5+1+0+1+2.5}{5} = \frac{7}{5} = 1.4.$$

If we calculate the cost of drunkenness from this, we get

$$\frac{ct(G)}{dct(G)} = \frac{3}{1.4} = \frac{15}{7} \approx 2.143.$$

6 Meyniel's conjecture

Another question in cops and robbers is whether we can provide a bound on the cop number of all connected graphs with n vertices, in terms of n. A very trivial one would just be $c(G) \leq n$, but we attempt to find something more useful that makes calculating the cop number easier. To formally explain what we mean by this bound, we give the following definition:

Definition 6.1. We write $f(x) = O(g(x))$ if there exists a positive real number k such that for all sufficiently large x,

$$f(x) \leq k \cdot g(x).$$

One of the most important open problems that arises from this definition is known as Meyniel's Conjecture.

Conjecture 6.2. *If G is a graph with n vertices, then $c(G) = O(\sqrt{n})$.*

In other words, it claims that for sufficiently large n, a graph with n vertices has cop number at most $k\sqrt{n}$, where k is some constant. If Meyniel's conjecture is true, it would be the tightest bound possible on the cop number because there are classes of graphs whose cop number approaches $k\sqrt{n}$ as n gets large. While Meyniel's conjecture has not yet been proven, weaker bounds have been. In the paper first stating the conjecture, an initial bound was provided: (See **[Fra87]**)

Theorem 6.3. *If G is a graph with n vertices then $c(G) = O\big(\frac{n \log \log n}{\log n}\big)$*

Before proving Theorem 6.3, we need to first give some definitions and then state another theorem.

Definition 6.4. In a graph $G = (V, E)$, the *degree* of a vertex u is the number of vertices $v \in V$ such that $(u, v) \in E$.

Definition 6.5. The *diameter* of a graph $G = (V, E)$ is

$$\max_{u,v \in V} (d(u, v)),$$

where $d(u, v)$ is the shortest distance between u and v.

Theorem 6.6. (Moore Bound) *Let G be a graph with n vertices, maximum degree $\Delta(G) > 2$ and diameter D. Then, we have*

$$n \leq 1 + \Delta\bigg(\frac{(\Delta(G) - 1)^D - 1}{\Delta(G) - 2}\bigg).$$

Proof. (of Theorem 6.3) Lemma 4.5 states that a path can be guarded by 1 cop. We also know that a vertex v and the set of its adjacent vertices can be guarded by 1 cop because of Proposition 4.2. From the Moore bound, we can determine that $n = O(\Delta^D)$, and eventually that both Δ and D cannot be less than $O\big(\frac{\log(n)}{\log(\log(n))}\big)$.

Since any path in G has at most D vertices and any vertex has at most degree Δ, there exists a subgraph X of G with at least $O(\frac{\log(n)}{\log\log(n)})$ vertices.

Now, consider the graph G', which is G with X deleted. If G' is disconnected, consider the connected component with the robber and move all the cops there. We now have

$$c(G) \leq c(G') + 1.$$

By induction, it follows that

$$c(G) = O(\frac{n \log\log n}{\log n}).$$

□

Currently, the best proven bound for all graphs with n vertices is as follows:

Theorem 6.7. *If G is a graph with n vertices then*

$$c(G) = O\left(\frac{n}{2^{(1-o(1))(\sqrt{\log_2 n})}}\right).$$

This was proven in **[SS11]** by Scott and Sudakov. The open problem of Meyniel's conjecture is only the latest and most famous facet of cops and robbers. There are variations such as active cops and robbers and limited cops and robbers, bounds on many different classes of graphs, and newly discovered unsolved problems. These involve other areas of mathematics and show that cops and robbers has many interesting explorations.

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