The Angel and Devil Problem

Tej Nadkarni and Aryan Agarwal

August 2021

1 Introduction

The angel and devil problem is a combinatorial game where 2 players, an angel and a devil, take turns maneuvering around an infinite chessboard. On the devil's turn, he can remove any square from the chessboard, and on the angel's turn, she can move to a range of squares limited in some way. The devil wins if he can put the angel in a position where she has no moves, and the angel wins if she can avoid such a position indefinitely. In this paper we discuss winning strategies for the players when we limit the maneuverability of the angel in various ways.

2 Problem Statement

An angel starts on some square of an infinite chessboard. Every turn, the angel can move to any square up to k squares in any direction (including diagonally) from where she currently is. We will call such an angel, an angel with power k, or a k-angel for short. After each of the angel's turns, the devil gets to remove one square from the board, so that the angel may no longer move there. The devil wins if the angel eventually gets trapped, and otherwise the angel wins. Who has a winning strategy for every value of k? What happens if we limit the angel's maneuverability in different ways?

3 Useful and Interesting Observations

Observation 3.1 The devil wins if he can build a wall around the angel with thickness k. Even if the angel can move around in the area enclosed by the wall, the angel cannot cross the wall, and the devil will eventually fill up the area enclosed by the wall until the angel has no moves.

Observation 3.2 The angel is not harmed when k is increased. This is because a winning strategy for the angel for $k = x$ will also work for $k = x + 1$.

Observation 3.3 A position is no better for the angel than the same position with an extra square removed. This is because the angel has no more and no better options than in the original position.

Observation 3.4 A k-angel can move to $(2k+1)^2 - 1$ squares on her turn (assuming none are blocked).

Definition 3.5: Let the Altered Angel and Devil Problem have the extra condition that the angel cannot go to a square she could have gone to on a previous turn. I.e. when the angel leaves a square, that square, and all of the $(2k+1)^2$ squares she could have moved to on that turn, are removed from the board (except for the square the angel lands on).

Theorem 3.6 For any value of k, the player with the winning strategy for the Angel and Devil Problem will also have a winning strategy for the Altered Angel and Devil Problem.

Proof. It is clearly impossible for the angel to win the altered problem if the devil wins the normal problem, since removing squares cannot help the angel. We will now prove that it is impossible for the devil to win the altered problem if the angel wins the normal problem. Let us assume on the contrary that the devil won the altered problem while the angel won the normal problem. This would imply that the angel's winning strategy for the normal problem would involve returning to at least one of the $(2k+1)^2$ squares. When the angel moves to a square she could have gone to on a previous turn, the devil will ignore everything that has happened since the angel was at the center of the $(2k+1)^2$ square, and pretend that the angel moved directly from the center of the square, to where the angel currently is. The devil would have blocked out extra squares by the time the angel reaches its current square, thus the devil will be strictly better off in this case than when the angel directly moves to the square she is currently on. Thus returning to such a square is sub-optimal, which is a contradiction. QED

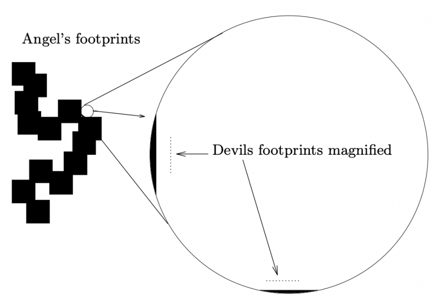

Figure 1: The area that a 1000-angel blocks every turn is much larger than the area the devil blocks. [1]

4 Case Where $k = 1$

Setup 4.1 First we will solve the problem for $k = 1$, which is the easiest case for this problem. The devil has the winning strategy. Let us note that the angel moves like a chess king. We, as the devil's advocate, must construct a wall with a thickness of 1 around the angel. The angel has the first move. After her move let us call the square that she is on, the origin i.e. $(0, 0)$. WLOG, let this square be white (the rest of the plane will be tiled like a normal chessboard). Let the square x units right, and y units up from the origin be (x, y). We will construct a wall on the perimeter of the square connected by the points $(-27, -27)$, $(27, -27)$, $(-27, 27)$, and $(27, 27)$. (We choose a 55×55 grid to make the proof easier, a 32×33 grid works as well.)

We will start by removing the following 20 squares: $-(27, -27)$, $(27, -27)$, $(-27, 27)$, $(27, 27)$, $(-26, -27)$, $(26, -27)$, $(-26, 27)$, $(26, 27)$, $(-27, -26)$, $(27, -26)$, $(-27, 26)$, $(27, 26)$, $(-25, -27)$, $(27, -25)$, $(-27, 25)$, $-27, -25$, $(25, 27)$, $(25, -27)$, $(-25, 27)$, $(27, 25)$. (The 5 squares, on the perimeter of the 55×55[1] square, closest to each corner are removed.) After we remove $(27, 25)$, and the angel moves, the angel must be within the square enclosed by the points $(-20, -20)$, $(20, -20)$, $(-20, 20)$, and $(20, 20)$. We will now use **Algorithm 4.2** to determine which square to remove.

Algorithm 4.2 If the angel is in the 49×49 square, we will remove[2] the closest[3] white square to the angel that is on the perimeter. If there are multiple such squares, we will remove the square that is furthest from another removed square. If there are still multiple such squares, we will remove a random one of these squares. We will continue this process until we run out of white squares in which case we will switch to black squares, or until the angel leaves the 49×49 square. If the angel ever wanders back into the 49×49 square, we will revert back to the algorithm described in this paragraph.

If the angel is not in the 49×49 square, we will remove the square (not necessarily white) closest to the angel. If there are multiple such squares, we will remove the square that is furthest from another removed square. If there are still multiple such squares, we will remove a random one of these squares. We will continue this until the wall is completed (and we win) or until the angel wanders back into the 49×49 square in which case we will revert back to the algorithm described in the previous paragraph. We will continue until we finish building the wall.

Theorem 4.3: Algorithm 4.2 is effective in trapping the angel.

[1] If an $n \times n$ square is mentioned without a specified center, assume the center to be the origin.

[2] Until a wall is fully constructed around the angel, we will not remove any squares that are not on the perimeter of the 55×55 square. This will always be a constraint on which squares we will remove.

[3] The distance between two squares is equal to $|x_1 - x_2| + |y_1 - y_2|$ where (x_1, y_1) and (x_2, y_2) are the coordinates of the squares.

Proof. The angel will escape the 55×55 square iff she can "fork"[4] two or more squares on the perimeter of the square. (This is why we remove the corner squares. If the angel is near the corner, five squares can be attacked at once.) Recall that after we remove $(27, 25)$, and the angel moves, the angel must be within the 41×41 square. This means that we will remove at least five white squares before the angel leaves the 49×49 square. This means that after the angel leaves the 49×49 square, all white squares on the perimeter, two units to each side of the angel, will be removed. On her next move, the only way the angel will be able to fork two squares on the perimeter is by forking two black squares. By removing the closest square (which will be black) we will prevent the angel from forking two such squares. No matter where the angel moves we will always be one step ahead (literally and figuratively - see the figure to understand one case of **Algorithm 4.2**). QED

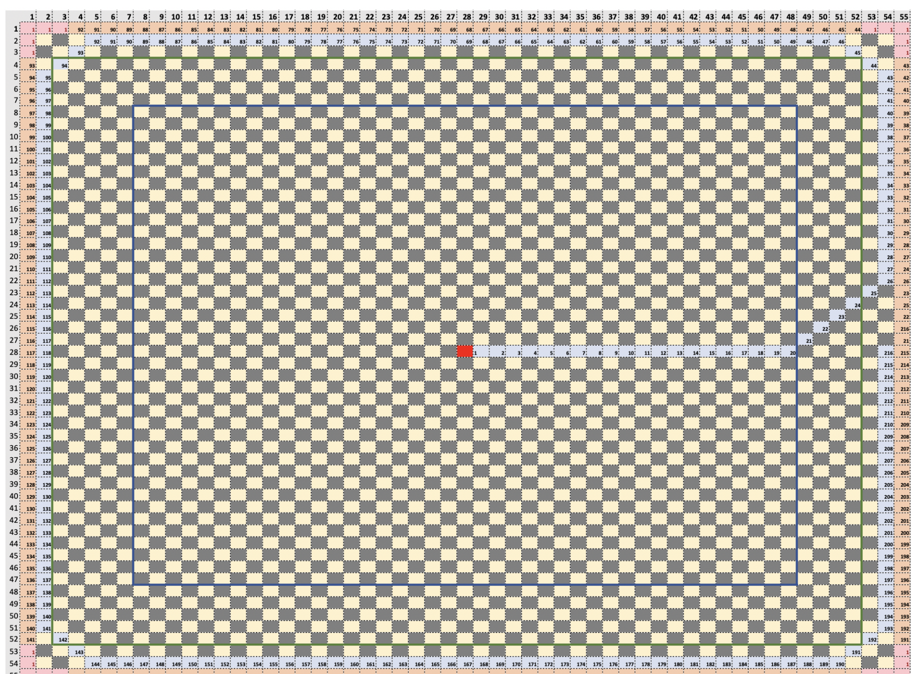

Figure 2: A visual depiction of *Algorithm 4.2*. Deep Red marks $(0, 0)$, light blue indicates moves of the Angel (along with move number), light red marks the first 20 moves of the Devil, and peach marks the subsequent moves of the Devil from 21 to 216. The Blue border indicates $40 * 40$ square, and the Green border indicates the $49 * 49$ square. The Angel moves first. After 216 moves, the wall is complete.

[4]In chess a fork is when a piece attacks two or more pieces at the same time.

5 Fools

The following games are all necessary proofs that show the complexity of the Angel's winning strategy. On the surface the Angel might seem much stronger than the devil, but common strategies can fall prey to macroscopic traps, as we will now see.

Definition 5.1 There are various types of fools. A generic fool is an angel with her abilities limited in some way.

Definition 5.2 A *k-fool* is a k-angel that increases her y-coordinate every turn.

Theorem 5.3 *The devil will beat a k-fool.*

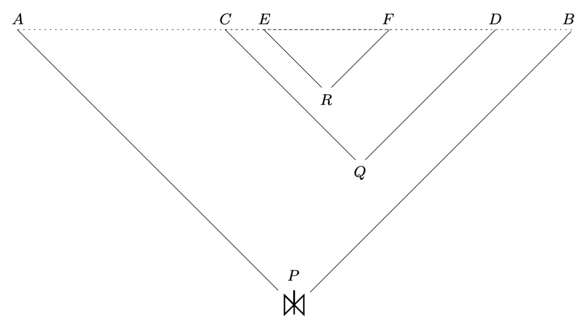

Figure 3: Capturing the k-fool [1]

Proof. Whenever the k-fool is at some point $P : P = (x_P, y_P)$, her future positions will be limited to the cone defined by all squares, (x, y), satisfying

$$(y - y_P) \geq \frac{\lceil |x - x_P| \rceil}{k}.$$

The devil will start by choosing an horizontal line that is a very large power of 2 above P. We will define the points where the cone intersects this line to be A and B. Let the height of the triangle formed be H. The devil will then remove 1 out of every $4k$ squares on the line AB so that the devil finishes by the time the fool is $\frac{H}{2}$ away from AB. The angel will now be at point Q, and be limited to a cone of half the size. The new cone will intersect line AB at points C and D. The devil will again remove 1 out of every $4k$ squares on line CD, so that the devil finishes by the time the fool is $\frac{H}{4}$ away from line CD. We will repeat this process $4k^2$ times until a wall of thickness k is constructed above the fool.

Definition 5.4 A *lax k-fool* is a k-angel that never decreases her y-coordinate.

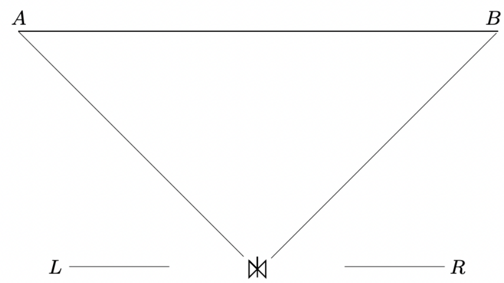

Figure 4: Catching the Lax k - fool [1]

Theorem 5.5 The devil will beat a *lax k-fool.*

Proof. The devil will use his odd moves to convert the lax k-fool into a normal fool of a greater power. The devil chooses two points, L and R, $4k^2$ units to the left and right of P, the starting position of the *lax k-fool.* As long as the *lax k-fool* stays on the first row, the devil will use his odd moves to alternately remove squares starting at R and L, moving left from R, and right from L. Thus we remove a square to the right and left of the *lax k-fool* every 4 turns. Thus in $4k$ turns we will have constructed a wall of thickness k to the left and right of the *lax k-fool.* If the *lax k-fool* continues staying on line LR then the *lax k-fool* will be forced to move upwards after $16k^2$ turns. Thus, after $16k^2$ moves the *lax k-fool* must move upwards. Since the *lax k-fool* must move upwards once every $16k^2$ moves, it is in essence a $16k^3$-fool. The devil will use his even moves to trap the $16k^3$-fool. QED

Definition 5.6: A *relaxed k-fool* (of laxity z) is an k-angel that does not decrease its *y-coordinate* by more than z, where z is some fixed arbitrary positive integer. Thus, if the *relaxed k-fool's* starting position is (x, y), and some time later its position is (X, Y), then $Y > y - z$.

Theorem 5.7: The devil can catch a *relaxed k-fool.*

Proof: The devil adapts his strategy of dealing with the *lax k-fool* to beat the *relaxed k-fool.* Let P be the starting position of the *relaxed k-fool.* Thus, we take a suitable distance D as before from P to the left (L) and right (R) of the *relaxed k-fool.* We choose D at such a distance that the devil has enough moves to create 2 rectangles of width k and length z in the negative y-direction. Thus, by the time these canyons are created, if the *relaxed k-fool* kept moving

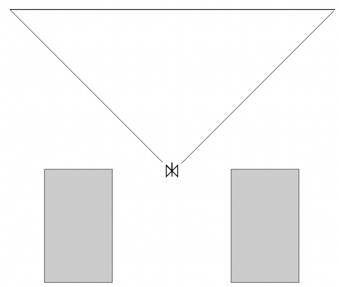

Figure 5: Catching the Relaxed Fool [1]

downwards, she is bound by the limit z we imposed in definition 5.6. Now, she has no choice but to increase her y-coordinate and cannot stay in the plane below P as she is bound by the 2 canyons the devil created, and will eventually be caught. After some moves, she will finally rise above P, and now the devil can now use the strategy for the *lax k-fool* to trap the *relaxed k-fool.* QED

Definition 5.8: Let the *Out-and-Out k-fool* be an k-angel who promises to strictly increase her distance from the origin (her starting position) on all moves.

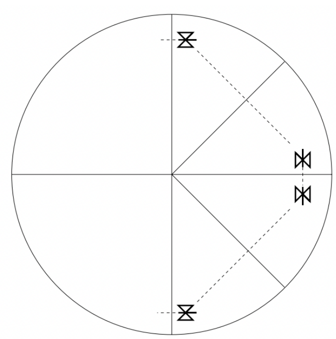

Figure 6: Kaleidoscope for capturing an Out-and-Out Fool [1]

Theorem 5.9: The devil can catch an *Out-and-Out k-fool.*

Proof: Before embarking upon this proof, let us discuss the board that this problem is played on. Is it necessary that we play on a chess board of side ∞? No. We can also play the game on a Euclidean plane where the Angel can "fly" to all points in a radius k while the Devil can eat any point from a unit disk in the plane. For example, if a k-angel is confined to a particular conical section, and promises to strictly increase her distance r from the Vertex O, we see that the cone transforms into a triangle, the r coordinate becomes the y coordinate, and the Devil's winning strategy for the k-fool applies.

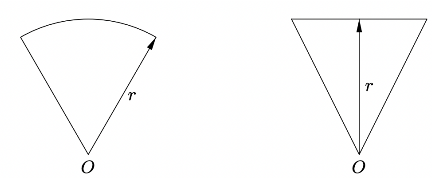

Figure 7: Transformation of Euclidean Plain regions to graph regions [1]

This is quite a powerful observation, as it helps us further generalize the solution to the Angel and Devil problem. Since the *Out-and-Out k-Fool* is capable of moving in all directions, we represent the situation as a graph with a circle with center $(0, 0)$ and radius a distance k as discussed in previous proofs. Thus, the devil has a winning strategy by dividing this circular plane into $2n$ sectors. If we imagine the circumference of the circle to be mirrors placed in a kaleidoscope, there $2n$ images, one in each sector. Thus, when the *Out-and-Out k-Fool* moves in any sector, we transform it into a triangle (as described above) and thus we can use the proof for the k-fool to win. There is some distortion caused due to transforming the conic section into a triangle, however we roughly find that an *Out-and-Out Fool* is of the form of a *$2nk$-fool*. QED

Definition 5.10: Let a *Relaxed Out-and-Out k-fool* be a *k-angel* that promises not to reduce her distance from the Origin by more than z, where z is some fixed arbitrary positive integer.

Theorem 5.11 The devil can beat a *Relaxed Out-and-Out k-fool*.

Proof: We can reduce a *Relaxed Out-and-Out k-fool* to a *relaxed k-fool* the same way we reduced the *Out-and-Out k-fool* to a plain *k-fool* above. Using the idea of transformation, we get the same inequality as before, albeit with slight distortions in strategy due to the transformation. Thus, if the Angel's starting position is (x, y) and its final position is (X, Y), then $Y > y - z$. This is a powerful proof and we will use this idea in proving an extremely important strategy for the devil: the Blass-Conway Diverting Strategy.

Thus, even complex strategies like the *Relaxed Fools* are easily beatable by the devil. This goes to show that any winning strategy for the Angel at $k \geq 2$ must be significantly more complex, and the Angel must follow a counter-intuitive strategy to win.

6 Blass-Conway Diverting Strategy

The Blass-Conway Diverting Strategy is a theorem that provides a very powerful strategy for the devil. However, it is not a conclusive proof for the case $k \geq 2$. It simply shows the strategies that are 'foolish' in nature must be avoided by the Angel if she is to win.

Theorem 6.1: There exists a *diversion strategy* for the Devil such that for each point P and each distance D in a plane, there will be a time $t_2 : t_2 > t_1$ such that the k-angel is D units nearer to P than she was at time t_1, for all P and D.

Proof: We have different combinations of P and D of the form: (P_0, D_0), (P_1, D_1), (P_2, D_2), (P_3, D_3), and so on. We can interpret such a game as a *Relaxed Out-and-Out k-fool* that promises not to go farther than D_n from P_n for all $n \in \mathbb{N}$. The devil can thus use his odd moves to trap the Angel when $n = 0$ as explained in **Theorem 5.11**. By applying the same theorem again, the Devil can use his moves, that are 2 times an odd number, on $n = 1$ to win. Thus, the devil can meet the n^{th} requirement by playing the strategy for the *Relaxed Out-and-Out k-fool* on his $(2^n \cdot O)^{th}$ moves, where O is an odd number.

This theorem thus generalizes all the approaches for the devil in the fool scenarios. Thus, The Blass Conway Diversion Strategy is the roadmap for the Devil to win the Angel Problem, if indeed he is winning.

7 Closing Remarks:

Through this paper, we have introduced the Angel Problem and described certain observations needed to find a general solution. We then utilised these observations to prove that the devil wins the $k = 1$ case, if he uses **Algorithm 4.2** as his strategy. In a pursuit to solve the general case of $k \geq 2$ we covered the different fools to show that various intuitive strategies for the Angel fail quite easily. We then used the proof of a *Relaxed Out-and-Out fool* to derive the *Blass Conway Diverting Strategy* - a general winning strategy for the devil. However, we saw that this theorem is not a conclusive proof for the general case $k \geq 2$ and only serves as a proof to show how the Devil can *potentially* win a game.

At this point the reader might be thinking that the devil is the clear victor of the game, however, the true power of the angel is vastly underestimated when we compute the outcome of the fools. Despite the multitude of theorems and examples in favour of the devil winning the Angel Problem, John Conway was still convinced that there is a winning strategy for the Angel when $k \geq 2$, and even offered a cash prize for whoever managed to solve the problem. Andras Mathe and Oddvar Kloster both separately offered proofs, showing that the angel wins $k \geq 2$.[2] Kloster described the winning strategy for the 2-angel to be to travel north as quickly as possible and detour around removed squares, if the added distance will not be more than twice that of the number of eaten squares avoided.

References

[1] John H Conway. The angel problem. *Games of no chance*, 29:3–12, 1996.

[2] Oddvar Kloster. A solution to the angel problem. *Theoretical Computer Science*, 389(1-2):152–161, 2007.

[3] Martin Kutz. *The Angel Problem, Positional Games, and Digraph Roots.* PhD thesis, PhD thesis, Freie Universität Berlin, 2004. http://www. diss. fu-berlin. de ..., 2004.

[4] Johan Wästlund. A weaker winning angel, 2008.