REDUCED CANONICAL FORM

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Abstract. We first define the basic of a game and some important notation, such as fuzzy games along with left and right stops. We then introduce cooling by $*$ and $*$ - projections, where the projection of G cooled by $*$ or $p(G_*)$ represents the reduced canonical form of G and look at an example. We look at some crucial properties of reduced canonical form, such as uniqueness and existence.

1. Introduction

Definition 1.1. We define a game G to be $\{G^L \mid G^R\}$, where G^L is the set of left options and G^R is the set of right options.

In other words, G^L consists of all the moves that Left can play and G^R consists of all the moves that Right can play. We define $\{|\}$ to be the empty position, which has a value of 0. However, the position $\{0 \mid 0\} = *$. Contrary to belief, $-G$ isn't $\{-G^L \mid -G^R\}$, but rather

$$-G = \{-G^R \mid -G^L\}.$$

We ask the reader to reference [Con00] for further rules in arithmetic of numbers and games, not necessarily all needed for the concepts introduced in this paper.

Definition 1.2. For games G and H, if $G - H \in \mathcal{N}$, then G is confused with H and G is considered to be a fuzzy game. We can write $G \| H$ to show that G is confused with H.

We also can state that $G \leq H$ meant that there doesn't exist an H^R such that $H^R \leq G$ and there also doesn't exist any G^L with $H \leq G^L$.

Definition 1.3. We have that

$$\mathbf{LS}(G) = \begin{cases} x_R & G \text{ is a number } x \\ \max(\mathbf{RS}(G^L)) & \text{otherwise} \end{cases}$$

and

$$\mathbf{RS}(G) = \begin{cases} x_L & G \text{ is a number } x \\ \min(\mathbf{LS}(G^R)) & \text{otherwise} \end{cases}.$$

We define $\mathbf{LS}(G)$ is the left stop of G and $\mathbf{RS}(G)$ is the right stop of G.

Definition 1.4. An infinitesimal is a game whose both left and right stops are zero.

Any arbitrary infinitesimal would be between negative and positive numbers. Additionally, the canonical form of a game is unique for all games and we can define it as the form with no dominated options or reversible moves. However, reduced canonical form, unlike canonical form, didn't need to have the same value as the initial form or be infinitesimally close to the original form. We only require the transformation from the game G to its reduced canonical form $\overline{G}$ to be linear.

2. Reduced Canonical Form

An important notion for constructing the reduced canonical form of a game is the operation of cooling. We will specifically work with cooling a game by $*$.

Definition 2.1. Let G_* to be the game G cooled by $\star$.

$$G_* = \begin{cases} G & \text{if } G \text{ was a number} \\ \{G_*^L + * \mid G_*^R + *\} & \text{otherwise.} \end{cases}$$

In this case, we say that G^L and G^R when cooled by $*$ are represented as G_*^L and G_*^R.

Using this, we see that if $G = H$ or even if $G - H$ is 0, then $G_* - H_*$ is just the zero game.

We can also define a $*-$ projection of a game H where H has the canonical form of $\{H_1 \mid H_2\}$ as $p(H)$ as:

$$p(H) = \begin{cases} x & \text{if } H = x \text{ or } x + *, \text{ where } x \text{ is a number,} \\ p\{(H_0^L) \mid p(H_0^R)\} & \text{otherwise.} \end{cases}$$

In fact, $p(G_*)$ was just the reduced canonical form of G.

Example. We consider the canonical form of a game X to be $X = \{\{3 \mid 0\}, 4\|0\}$. We can try and evaluate this using the fact that $p(X_*)$ was the reduced canonical form of X. However, we should first address what the $\|$ meant in this expression, so we consider another situation.

Consider the game $Y = 8\|5 \mid 2|||7 \mid -4\|-3 \mid -17$. The double and triple slashes only indicate that the precedence is higher. This in fact is just the canonical form

$$Y = \{\{8 \mid \{5 \mid 2\}\} \mid \{\{7 \mid -4\} \mid \{-3 \mid -17\}\}\}.$$

Leaving the brackets and rather using multiple slashes seems to be much more convenient and we therefore stick to this convention for the this paper.

We now return to our earlier example for finding the reduced canonical form of X, which was as we defined it, $\{\{3 \mid 0\}, 4\|0\}$. When X is cooled by $*$, we get

$$X_* = \{\{3 \mid 0\}_* + *, 4 + *\|*\} = \{\{3 + * \mid *\} + *, 4 + *\|*\}.$$

To simplify this further, we use the following theorem from [BCG], which dictates how to work with a game added to $*$:

Theorem 2.2. *An arbitrary game* $P = \{P^L \mid P^R\}$ *added to* $*$ *is just*

$$P + * = \{P^L \mid P^R\} + * = \{P^L + * \mid P^R + *\} = \{P^L * \mid P^R *\}.$$

We can use this to find X_*, which as before, was

$$X_* = \{\{3 + * \mid *\} + *, 4 + * \| *\}.$$

So

$$\{3 + * \mid *\} + * = \{3 + * + * \mid * + *\}$$

and since we also know that $* + * = 0$, we can simplify this further to get that $\{3 + * \mid *\} + * = \{3 \mid 0\}$.

Therefore,

$$X_* = \{\{3 + * \mid *\} + *, 4 + * \| *\} = \{\{3 \mid 0\}, 4* \| *\}.$$

The Left option of $\{2 \mid 0\}$ is reversible and can be easily replaced with the other options in X. Moreover, we have that X_* just simplifies to $\{4* \mid *\}$ and this since this has no reversible moves so this must be the canonical form of X_*.

Finding the projection of X_* or $p(X_*)$ gives

$$p(X_*) = \{p(4*) \mid p(*)\} = \{4 \mid 0\}.$$

This seems to be much more concise and easier to understand than the original form of X. It also makes this much easier to work with for different operations and possible interpretations.

3. Properties of Reduced Canonical Form

Before we introduce some key properties of the reduced canonical form of a number, we need to define some crucial terms.

As we had previously defined, an infinitesimal is a game with both left and right stops equal to zero, where an infinitesimal is strictly in between all positive and numbers. In other words, G and H are infinitesimally close or $G - H$ is infinitesimal, which means that

$$x \geq G - H \geq -x,$$

for any positive x. We could also write that $G \equiv H \pmod{\text{Inf}}$ if G and H are infinitesimally close.

Definition 3.1. We write $G \geqq H \pmod{\text{Inf}}$ to say that to emphasize that the inequality was under (mod Inf).

Definition 3.2. We say that a game is hot if and only if $\mathbf{LS}(G) > \mathbf{RS}(G)$.

Finally, we define another set of terms to prove particular properties of reduced canonical form of a number.

Definition 3.3. If G is a short game, then we have the following:

(a) A Left option G^{L_1} is Inf-dominated (by G^{L_2}) if $G^{L_2} \geqq G^{L}$, for some other Left option G^{L_2}.

(b) A Right option G^{R_1} is Inf-dominated (by G^{R_2}) if $G^{R_2} \leqq G^{R_1}$ for some other Right option G^{R_2}.

(c) A Left option G^{L_1} is Inf-reversible (through $G^{L_1R_1}$) if $G^{L_1R_1} \leqq G$ for some Right option $G^{L_1R_1}$.

(d) A Right option G^{R_1} is Inf-reversible (through $G^{R_1L_1}$) if $G^{R_1L_1 \geqq G}$ for some Left option $G^{L_1R_1}$.

The use of these Inf-dominated and Inf-reversible options will be essential for the next definitions for reduced canonical form. In fact, a game is in canonical form if the game is hot and has neither Inf-dominated nor Inf-reversible options. We will look at this idea further in Definition 3.5.

Definition 3.4. H is a subposition of a game G if there exists a sequence of moves leading from G to H.

Using these definitions, we are finally ready to define the main property of a game in reduced canonical form and following theorems from it.

Definition 3.5. Let Q be a short game; it's in reduced canonical form if for every subposition H of Q, one of the following are true:

(i) H is a number in canonical form

(ii) H is hot and contains no Inf-dominated or Inf-reversible options.

Using these ideas, we introduce a lemma regarding the replacement of infinitesimals.

Lemma 3.6. *If G is not equal to a number, and G' is created by replacing the Left option of G^L with $G^{L'}$, then we state that if $G^{L'} \geqq G^L$, then $G' \geqq G$ must be true. However, if $G^{L'} \leqq G^L$, then $G' \leqq G$ must be true.*

Proof. Suppose that $G^{L'} \geqq G^L$ and so we must show that $G' - G + x$ is a game in which Left would win by playing second for any positive x. We also see that $-G$ isn't a number, but x is. So we can ignore Right's beginning moves on x, by the Number Avoidance Theorem. Additionally, Right's moves have similar responses to the other components, except for when Right plays to $G' - G^L + x$.

However, the response is positive since $G^{L'} - G^L + x$, which is true because $G^{L'} \geqq G^L$. Remaining moves of G' and $-G$ have similar responses to the other component as well. A similar argument follows if $G^{L'} \leqq G^L$, then $G' \leqq G$. ■

We see that for some games, there must be a reduced canonical form to the more original form. However, we don't know whether all games have reduced canonical forms and we therefore introduce the following theorem, helping us to prove the main one.

Theorem 3.7. *If G is hot, then we assume that some Left option G^{L_1} is Inf-reversible through $G^{L_1R_1}$. If*

$$G' = \{G^{L_1R_1L}, G^{L'} \mid G^R\},$$

then $G' \equiv G$ must be true.

Proof. To prove this, we simply have to show that G' isn't a number. Since G is a hot game, the left stop must be greater than the right stop or that $\mathbf{LS}(G) > \mathbf{RS}(G)$. We can prove that $\mathbf{LS}(G') > \mathbf{LS}(G)$ and $\mathbf{RS}(G') < \mathbf{RS}(G)$ to prove that G' is also a hot game.

We then choose some G^R where $\mathbf{LS}(G^R) = \mathbf{RS}(G)$. G^R therefore also must be the right option of G' and we get that $\mathbf{LS}(G^R) \geq \mathbf{RS}(G')$ since we also know that $\mathbf{RS}(G^L) < \mathbf{LS}(G)$ for every G^L and $\mathbf{LS}(GR) > \mathbf{RS}(G)$ for every G^R, even if G is equal to a number. From this we can state that $\mathbf{RS}(G') \leq \mathbf{RS}(G)$ and for $\mathbf{LS}(G')\mathbf{LS}(G)$, there are 2 cases.

The first case is where $\mathbf{RS}(G^{L_1}) = \mathbf{LS}(G)$, which then means that $\mathbf{LS}(G^{L_1R_1}) \leq \mathbf{RS}(G)$. Therefore, we have that

$$\mathbf{LS}(G^{L_1R_1}) \geq \mathbf{LS}(G) > \mathbf{RS}(G) \geq \mathbf{RS}(G^{L_1R_1}),$$

which means that $G^{L_1R_1}$ isn't a number. There then would be some $G^{L_1R_1L}$ where the right stop of this game s equivalent to the left stop of $G^{L_1R_1}$, which is greater than the left stop of G. In other words, we have that

$$\mathbf{LS}(G^{L_1R_1}) \geq \mathbf{LS}(G) > \mathbf{RS}(G) \geq \mathbf{RS}(G^{L_1R_1})$$

which means that $G^{L_1R_1}$ is not a number.

Moreover, there must then be some $G^{L_1R_1L}$ with

$$\mathbf{RS}(G^{L_1R_1L}) = \mathbf{LS}(G^{L_1R_1}) \geq \mathbf{LS}(G).$$

Additionally, $G^{L_1R_1L}$ is the Left option of G', which means that $\mathbf{LS}(G') \geq \mathbf{LS}(G)$.

We now can consider the second case of where $\mathbf{RS}(G^{L_1}) \neq \mathbf{LS}(G)$ so we have $\mathbf{RS}(G^{L'}) = \mathbf{LS}(G)$ for some $G^{L'}$ but $G^{L'}$ is also a left option of G'. In either of these cases, we have that $\mathbf{LS}(G') \geq \mathbf{LS}(G)$.

■

We now turn our attention towards the idea of existence of reduced canonical form of a game, thus introducing the following theorem.

Theorem 3.8. *For every short game G, there exists H such that $H \equiv G$ in reduced canonical form.*

Proof. We see that K must be the canonical form of a number x if the game G is infinitesimally close to a number x. If G is a hot game, then by induction, the theorem must be true for all options of G. By Lemma 3.6, no proper subposition of G have any Inf-dominated or Inf-reversible options. If this statement holds for G, we have proven the statement, as desired.

However, if G has an Inf-dominated option, then by Theorem 3.7, it can be eliminated, resulting in a simpler game of $G' - G$. By induction, there must exist a $K - G'$ that's in reduced canonical form. An identical argument follows when G has a Inf-reversible option, which can be bypassed similarly, also using Theorem 3.7. ■

For other properties and proofs of reduced canonical form and hot games in general, we ask the reader to reference [Sie13].

References

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