Nim With Many Players

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Abstract

We discuss the n-player versions of the games Nim and Nim_k equipped with the podium rule. Much of this paper comes from [1]. Some material from Chapter 3 of [2] is reviewed as it becomes relevant in proving new results.

1 Introduction

We traditionally only study combinatorial games with two players because the theory is more elegant. This is because, in games with more players, there often arise situations in which a group of players can form a coalition to prevent other players from winning. However, the formation of these groups as well as their choice of who to prevent winning is arbitrary, which does not lend itself well to an analysis. For instance, problem 1.16 in [2] shows us that in a three player version of the subtraction game Matches, there is a situation in which the first player may choose which of the other two win. But who is to say what their choice should be? To fix these issues, we shall modify our definitions of combinatorial games (in particular, what a winning condition is in these games) to accommodate more than two players.

2 Two Player Nim

A normal-play Nim game is a set of piles of stones (the piles are not necessarily the same height). Two players alternate moves, and on their turn they can remove any non-zero number of stones from a single pile. The player who finds themselves with no moves available to them on their turn to play loses. We shall write a Nim game with n piles of height $a_1, a_2, \ldots, a_n$ as

$$[a_1, a_2, \ldots, a_n].$$

Nim is an example of an impartial game - one in which all players have the same set of moves available to them in any position. For impartial games, we call a game a $\mathcal{P}$ position if the player to start loses, or an $\mathcal{N}$ position if the player to start wins. The following result, called the Partition Theorem in [2], is essential to the proof of Theorem 2.2. We will later generalize it to prove similar results about n-player Nim.

Theorem 2.1 *Let $\mathscr{S}$ be a set of finite impartial games closed under sub-positions, and let $\mathscr{P}$ and $\mathscr{N}$ partition it. Suppose that for every game in $\mathscr{N}$ there exists a move to a game in $\mathscr{P}$, and for every game in $\mathscr{P}$ all moves are to games in $\mathscr{N}$. Then we have $\mathscr{N} = \mathcal{N}$ and $\mathscr{P} = \mathcal{P}$.*

Proof. We will show this by induction on the height of the game tree of games in $\mathscr{S}$. Firstly, it must be the case that all terminal positions are contained in $\mathscr{P}$, since they cannot be in $\mathscr{N}$ because there are no moves from them (in particular, there does not exist a move to a game in $\mathscr{P}$). These terminal positions are also in $\mathcal{P}$. Now, any game with a terminal position as an option must be in $\mathscr{N}$, and they are also in $\mathcal{N}$. Inductively, we will place games whose options are all in $\mathscr{N}$ into $\mathscr{P}$, observing that these games are also in $\mathcal{P}$. Similarly, we will place games who have at least one option that is a game in $\mathscr{P}$ into the set $\mathscr{N}$, observing that these games are also in $\mathcal{N}$. Since $\mathscr{P}$ and $\mathscr{N}$ partition $\mathscr{S}$ by definition, this process will cover all games. Thus, $\mathscr{P} = \mathcal{P}$ and $\mathscr{N} = \mathcal{N}$.

In 1902, Bouton published Theorem 2.2 telling us the $\mathcal{P}$ positions of Nim, effectively solving the game. Before presenting it, we must first introduce the nim-sum operation. To compute the nim-sum of two binary numbers a and b, written as $a \oplus b$, we concatenate their digit-wise sums modulo 2. Thus

$$1001 \oplus 1101 = 0100.$$

For numbers written in a different base, we first write them in binary and then compute their nim-sum normally. The operation is commutative and associative, so we may 'add' more than two integers freely.

Theorem 2.2 *The Nim game $[a_1, a_2, \dots, a_n]$ is a $\mathcal{P}$ position if and only if*

$$a_1 \oplus a_2 \oplus \cdots \oplus a_n = 0. \tag{1}$$

Proof. Let $\mathscr{S}$ be the set of Nim games, and suppose $\mathscr{P}$ is the subset of $\mathscr{S}$ where (1) is true for each of its elements and $\mathscr{N}$ is the set of all other games in $\mathscr{S}$. If $G = [a_1, a_2, \dots, a_n] \in \mathscr{P}$ let us suppose, without a loss of generality, that the first player moves in a_1 to a_1'. Since $a_1' \neq a_1$ we must have

$$a_1' \oplus a_2 \oplus \cdots \oplus a_n \neq a_1 \oplus a_2 \cdots \oplus a_n = 0,$$

so that all moves from games in $\mathscr{P}$ are to games in $\mathscr{N}$. Conversely, if $G \in \mathscr{N}$ then we have

$$a = a_1 \oplus a_2 \oplus \cdots \oplus a_n \neq 0.$$

Let us consider the leftmost digit of a which is non-zero; say, in position j. Since $a \neq 0$, there are an odd number of piles with a 1 in the j-th digit of their binary representations. In particular, there is one such pile which, without a loss of generality, we will assume to a_1. We claim that the move that replaces a_1 with $a \oplus a_1$ makes the game a $\mathscr{P}$ position. Firstly, since in at least the j-th digit, the number $a \oplus a_1$ has a 0 where a_1 has a 1, this move is possible (we are reducing the number of stones in the pile). Performing this move, the game becomes

$$a \oplus a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_n = a \oplus a = 0,$$

which is in $\mathscr{P}$. Then, by Theorem 2.1, we have $\mathscr{N} = \mathcal{N}$ and $\mathscr{P} = \mathcal{P}$ as required.

A more general version of Nim is the game Nim_k, in which a player can remove stones from up to k piles on their move. Thus, Nim is the same as Nim_1. Theorem 2.2 can be extended to describe Nim_k, but for this we require a generalization of the nim-sum, called the q-nim-sum. To compute the q-nim-sum of two binary numbers a and b, written as $(a \oplus b)_q$, we concatenate their digit-wise sums modulo q. Thus

$$(1001 \oplus 1101 \oplus 0101)_3 = 2200.$$

Similarly, for numbers written in different bases, we first write them in binary and then compute their q-nim-sum normally.

Theorem 2.3 *The Nim_k game $[a_1, a_2, \ldots, a_n]$ is a $\mathcal{P}$ position if and only if*

$$(a_1 \oplus a_2 \oplus \cdots \oplus a_n)_{k+1} = 0.$$

We omit the proof as it arises as a special case of Theorem 3.4 in the next section.

3 N-Player Nim

We begin with an example to illustrate the necessity of a change in our definition of winning and losing in the context of an n-player game. Suppose that Alice, Bob and Charlie play (in that order) the Nim game $[1, 2]$. Then Alice may move to one of $[1, 0], [0, 2]$ or $[1, 1]$. If she moves to one of the first two then Bob is guaranteed a win, but if she moves to $[1, 1]$ then Charlie is guaranteed a win. Thus, Alice cannot win but may choose which one of Bob or Charlie does. However, we cannot say what her choice should be. To rectify this, we will change our notion of winning using what is called the *podium rule.*

First, some conventions. In an n-player version of Nim, we begin by naming the players $P_1, P_2, \ldots, P_{n-1}$ and P_n. They rotate turns, so that P_1 plays first, then P_2, and so on until P_1 plays again after P_n (if they have a move). The game will end when one of the players cannot make a move on their turn to play. At this point, we assign each of them a rank in the following way.

Suppose that the game ends on P_i's turn to play so that they are the first player with no moves available to them. Then, we rank the players in increasing order as $P_i, P_{i+1}, \ldots, P_n, P_1, \ldots, P_{i-1}$.

The key assumption we shall now make is that each player will adopt a strategy that maximizes their individual rank. This is what removes the possibility of coalitions. For instance, in the example at the beginning of this section, it can be shown that the ranking upon playing either of the first two moves is P_3, P_1, P_2 and the ranking upon playing the third move is P_1, P_2, P_3. Since P_1 (that is, Alice) will now always play to maximize their individual rank, she will choose to make the move to $[1, 0]$ (or $[0, 2]$ - it does not matter).

Note that the last player to make a move obtains the highest rank, so we shall say that the game G is an $i - 1$ position if P_i ends the game (we discard the notion of a $\mathcal{P}$ and $\mathcal{N}$ position). We call this the outcome of G, denoted by $o(G)$. We can summarize the optimal strategy stated above by saying that if a player has the set of moves $\{G_1, G_2, \ldots, G_m\}$ available to them, they should move to G_k if

$$o(G_k) = \min \{o(G_1), o(G_2), \ldots, o(G_m)\}.$$

For the proof of the main result of two player Nim, we used the Partition Theorem. To prove similar results now, we require a generalization of it in the context of n-player Nim.

Theorem 3.1 *There are two parts:*

(a) An n player Nim game G is a $k > 0$ position if and only if we can reach a 0 position in less than k moves, and k is the smallest such integer for which this is true.

(b) Otherwise, G is a 0 position; that is, G is a 0 position if and only if we cannot reach another 0 position in less than n moves.

Proof. We prove only the forward directions; the converses are true almost by definition.

(a) Suppose that G was a k position yet we could not reach a 0 position in less than k moves. But then P_{k+1} cannot end the game, since they cannot play in a 0 position. Note that the requirement that k be minimal is clear.

(b) Suppose that G is a 0 position yet we could reach another 0 position in less than n moves; say, in j moves where $0 < j < n$. But then P_{j+1} ends the game, so that G is a j position.

We now aim to prove a straightforward test to determine whether an n player Nim game is a 0 position. If a game G is not a 0 position, then Theorem 3.1 tells us that we can test sub-positions of G to see whether they are 0 positions, and that such a sub-position is less than n moves away. Together, these provide a reasonable method for determining the outcome of a given Nim game.

Theorem 3.2 *Suppose $G = [a_1, a_2, \ldots, a_k]$ is an n-player Nim game. Then G is a 0 position if and only if*

$$(a_1 \oplus a_2 \oplus \cdots \oplus a_k)_n = 0.$$

Proof. Set $(G) = (a_1 \oplus a_2 \oplus \cdots \oplus a_k)_n$. Let $L(G)$ be the leftmost non-zero digit of (G) in base n; if (G) is 0 then set $L(G) = 0$ as well.

We begin by showing the reverse direction; that is, if $(G) = 0$ then G is a 0 position. By Theorem 3.1, this is the same as showing that we cannot reach a 0 position in less than n moves. Consider a sub-position G' of G. If $L(G) \neq 0$ then $L(G') \geq L(G) - 1$. Further, if $L(G) = 0$ then $L(G') = n - 1$. Thus, by induction, our claim is true.

Conversely, we now show that if G is a 0 position, then $(G) = 0$. Equivalently, we shall show that if $(G) \neq 0$ then G is not a 0 position. By Theorem 3.1, this is the same as showing that we can reach a 0 position in less than n moves. The proof is involved, so it will be presented as its own lemma below (this is Lemma 1 in [1]).

Lemma 3.3 *Let n be a positive integer and let $a_1, a_2, \ldots, a_m$ be non-negative integers. Write each a_i in binary as $a_{i1}a_{i2}\ldots a_{it}$ (here t will depend on i, of course). Then let s be the smallest integer so that*

$$\sum_{i=1}^{m} a_{is}$$

is not divisible by n. Now assume that the binary number $a_{is}a_{i(s+1)}\ldots a_{it} = 11\ldots1$ for all $i \leq k$, where k is a suitable non-negative integer smaller than m. Then we will prove the existence of non-negative integers $b_1, b_2, \ldots, b_m$ such that:

1. *$b_i \leq a_i$ for all $i \in \{1, 2, \ldots, m\}$*
2. *If b_i may be written in binary as $b_{i1}b_{i2}\ldots b_{it}$ then the sum*

   $$\sum_{i=1}^{m} b_{iq}$$

   is divisible by n for all $q \in \{1, 2, \ldots, t\}$.
3. *There is a permutation π on the set $\{1, \ldots, m\}$ which fixes $\{1, \ldots, k\}$ so that $b_{\pi(i)} = a_{\pi(i)}$ for all $i \geq n$.*

Proof. We prove the lemma by induction on the quantity $t - s$. Let r be the remainder when

$$\sum_{i=1}^{m} a_{is}$$

is divided by n. Up to a reordering of the integers $a_{k+1}, \ldots, a_m$, we can assume that $a_{is} = 1$ for all $i \leq r$. Now, for $i \leq r$, define $\overline{a_i}$ to be the number whose binary representation is $a_{i1}a_{i2}\ldots a_{i(s-1)}011\ldots1$, where there are $t-s$ copies of

1 at the end. For $i > r$, set $\overline{a_i} = a_i$. Thus, we can say $\overline{a_i} \leq a_i$ for all i. We shall also write $\overline{c_{i1}c_{i2}\dots c_{it}}$ for the binary representation of c_i. Now, if we have

$$\sum_{i=1}^{m} \overline{c_{iq}}$$

to be divisible by n for each $q \in \{1, 2, \dots, t\}$ then we can set $b_i = \overline{a_i}$ to complete the proof. Otherwise, set $\overline{s}$ to be the smallest integer such that

$$\sum_{i=1}^{m} \overline{a_{i\overline{s}}}$$

is not divisible by n. We must have $\overline{s} > s$. Further, $\overline{a_{i\overline{s}}a_{i(\overline{s}+1)}\dots a_{it}} = 11\dots1$ for all $i \leq k$. Up to a reordering of $\overline{c_{k+1}\dots c_m}$ there exists $\overline{k}$ such that $\overline{k} > k$ and $\overline{c_{i1}c_{i2}\dots c_{it}} = 11\dots1$ if and only if $i \leq \overline{k}$. Finally, using the inequalities $\overline{c_i} \leq c_i$ and $\overline{k} > k$ along with the induction hypothesis on $t - s$ we complete the induction.

The claim follows directly from this lemma.

As an example, consider the four player Nim game $[14, 13, 11, 8, 7, 4, 3]$. It is easily checked that

$$(14 \oplus 13 \oplus 11 \oplus 8 \oplus 7 \oplus 4 \oplus 3)_4 = 0,$$

so that the game is a 0 position. Thus, the final ranking will be P_1, P_2, P_3, P_4.

We conclude with a similar theorem about the game Nim_k.

Theorem 3.4 *Suppose $G = [a_1, a_2, \dots, a_m]$ is an n-player Nim_k game. Then G is a 0 position if and only if*

$$(a_1 \oplus a_2 \oplus \cdots \oplus a_m)_{nk-k+1} = 0.$$

Proof. We will use the same notation we did in the proof of Theorem 3.2. Consider a sub-position G' of G. If $L(G) \neq 0$ then $L(G') \geq L(G) - k$. Further, if $L(G) = 0$ then $L(G') \geq (n-2)k + 1$. The proof will now be identical to that of Theorem 3.2; using the inequalities to prove one direction, and Lemma 3.3 to prove the other.

Note that when $n = 2$, the statements of Theorem 3.2 and Theorem 3.4 match their counterparts in Section 2 as we would expect.

References

[1] S.Y.R. Li. N-person nim and n-person moore's games. *International Journal of Game Theory*, 7:31–36, 1978.

[2] Simon Rubinstein-Salzedo. *Combinatorial Game Theory.*