BIDDING GAMES

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ABSTRACT. In this paper, we cover another class of games known as bidding games in which the players have to bid for the right to make the next move.

1. Introduction

There are two game theories. The first is referred to as matrix game theory. In matrix games, two players make simultaneous moves and a payment is made from one player to the other depending on the chosen moves. Optimal strategies often involve randomness and concealment of information.
The other game theory is combinatorial game theory. In combinatorial games, two players move alternately. We may assume that each move consists of sliding a token from one vertex to another along an arc in a directed graph. A player who cannot move loses. There is no hidden information and there exist deterministic optimal strategies.

1.1. **Game proposed by David Richman.** In the 1980's, David Richman suggested a class of games that share some aspects of both types of game theory. The setup of the game is as follows:
The game is played by two players, Mr.Blue and Ms.Red, each of whom has some money. There is an underlying combinatorial game in which a token rests on a vertex of some finite directed graph. There are two special vertices denoted by b and r. Blue's goal is to bring the token to b and Red's goal is to bring the token to r. The two players repeatedly bid for the right to make the next move. Each player writes a non-negative real number(lesser than the number of dollars he or she has) on a card. The player with the higher bid on that turn gets to move the token from the vertex it currently occupies along the arc of the directed graph to a successor vertex. The winner of the bid also has to pay the amount of the bid to the opponent in order to make the move. If the two bids are equal, the tie is broken by flipping a coin. The game terminates when one player moves the token to their respective target vertices. Since money loses all value at the end of the game, the sole objective of the player is to make the token reach the appropriate vertex. The game is a draw if neither vertex is ever reached.

2. The Richman Cost Function

For the entirety of this paper, D denotes a directed graph (V, E) with a distinguished blue vertex b and a distinguished red vertex r. All other vertices are considered to be colored black. We assume that there is a path to either r or b from every vertex. We also assume that every vertex has a finite number of successors. Since the monetary transactions stay between the two players throughout the game, the total money supply remains fixed. For

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convenience, let's assume the total money supply to be equal to a dollar. The money is infinitely divisible.

2.1. **The Concept of Costs.** For $v \in V$, let $S(v)$ denote the set of successors of v in D, that is, $S(v) = \{w \in V : (v, w) \in E\}$. Given any function $f : V \to [0, 1]$, we define

$$f^+(v) = max_{w \in S(v)} f(w) \quad and \quad f^-(v) = min_{u \in S(v)} f(u).$$

The key to playing this Richman game on D is to attribute costs to the vertices of D such that the cost of every vertex (except r and b) is the average of the lowest and highest cost of its successors.

Definition 1. A function $R : V \to [0, 1]$ is called a Richman Cost Function if

$$R(b) = 0, \qquad R(r) = 1$$

and for every other $v \in V$ (black colored vertices), we have

$$R(v) = \frac{1}{2}(R^+(v) + R^-(v))$$

2.2. **Theorems:**

Theorem 2. *There exists a Richman Cost Function R(v) for any directed graph D.*

Proof. We introduce a function *R(v,t)*. Let *R(b,t)* $= 0$ and *R(r,t)* $= 1$ for all $t \in \mathbb{N}$. For $v \notin \{b, r\}$, we define $R(v, 0) = 1$ and

$$R(v, t) = \frac{1}{2}(R^+(v, t - 1) + R^-(v, t - 1)$$

for $t > 0$. It is easy to see that $R(v, 1) \leqslant R(v, 0)$ for all v, and a simple induction shows that *R(v,t+1)* $\leqslant$ *R(v,t)* for all v and all $t \geqslant 0$. Therefore *R(v,t)* is weakly decreasing and bounded below by zero as $t \to \infty$, hence convergent. It is also evident that function $v \mapsto \lim_{t\to\infty}$ *R(v,t)* satisfies the definition of a Richman function. ■

Theorem 3. *Suppose Blue and Red play the Richman Game on the digraph D with the token initially located at vertex v. If Blue's share of the total money exceeds R(v)*$= \lim_{t\to\infty} R(v, t)$, *Blue has a winning strategy. Indeed, if his share of the money exceeds R(v,t), his victory requires at most t moves.*

Proof. Without loss of generality, money can be scaled so that the total supply is one dollar. Whenever Blue has over *R(v)* dollars, he must have over *R(v,t)* dollars for some *t*. We prove the claim by induction on t. At t = 0, Blue has over *R(v,0)* dollars only if $v = b$, in which case he has already won.

Now assume the claim is true for $t - 1$, and let Blue have more than *R(v,t)* dollars. There exist neighbours u and w of v such that $R(u, t-1) = R^-(v, t-1)$ and $R(w, t-1) = R^+(v, t-1)$, so that $R(v, t) = \frac{1}{2}(R(w, t - 1) + R(u, t - 1))$. Blue can bid $\frac{1}{2}(R(w, t - 1) - R(u, t - 1))$ dollars. If Blue wins the bid at v, then he moves to u and forces a win in at most $t - 1$ moves(by the induction hypothesis), since he has more than $\frac{1}{2}(R(w, t - 1) + R(u, t - 1))$-$\frac{1}{2}(R(w, t-1) - R(u, t-1))$=$R(u, t-1)$ dollars left. If Blue loses the bid, then Red will move to some z, but Blue now has over $\frac{1}{2}(R(w, t - 1) + R(u, t - 1))$+$\frac{1}{2}(R(w, t - 1) - R(u, t - 1))$= $R(w, t - 1) \geqslant R(z, t - 1)$ dollars, and again wins by the induction hypothesis. ■

Before moving on to the next theorem, we require the following definition and technical lemma.

Definition 4. An edge is said to be an edge of steepest descent if $R(u) = R^{-}(v)$. Let $\bar{v}$ be the transitive closure of v under the steepest-descent relation. That is, $w \in \bar{v}$ if there exists a path $v = v_0, v_1, v_2, ..., v_k = w$ such that (v_i, v_{i+1}) is an edge of steepest descent for $i = 0, 1, ..., k-1$.

Lemma 5. *Let R be any Richman cost function of the digraph D. If $R(z) < 1$, then $\bar{z}$ contains* b.

Proof. Suppose $R(z) < 1$. Choose $v \in \bar{z}$ such that $R(v) = min_{u \in \bar{z}} R(u)$. Such a v must exist because D (and hence $\bar{z}$) is finite. If $v = b$, we are done. Otherwise, assume $v \neq b$, and let u be any successor of v. The defintion of v implies $R^{-}(v) = R(v)$, which forces $R^{+}(v) = R(v)$. Since $R(u)$ lies between $R^{-}(v)$ and $R^{+}(v)$, $R(u) = R(v) = R^{-}(v)$. Hence (v, u) is an edge of steep descent, so $u \in \bar{z}$. Moreover, u satisfies the same defining property that v did (it minimised R(.) in the set $\bar{z}$), so the same proof shows that for any successor w of u, $R(w) = R(u)$ and $w \in \bar{z}$. Repeating this, we see that for any point w that may be reached from v, $R(w) = R(v)$ and $w \in \bar{z}$. On the other hand, $R(r)$ is not equal to $R(v)$ (since $R(v) \leqslant R(z) < 1 = R(r)$), so r cannot be reached from v. Therefore b can be reached from v, so we must have $b \in \bar{z}$. ∎

Theorem 6. *If the directed graph D is finite, then there is only one Richman cost function on D.*

Proof. Suppose that R_1 and R_2 are Richman cost functions of D. Choose v such that $R_1 - R_2$ is maximised at v; such a v exists since D is finite. Let $M = R_1(v) - R_2(v)$. Choose u_1, w_1, u_2, w_2 (all successors of v) such that $R_i^{-}(v) = R(u_i)$ and $R_i^{+}(v) = R(w_i)$. Since $R_1(u_1) \leqslant R_1(u_2)$, we have

$$R_1(u_1) - R_2(u_2) \leqslant R_1(u_2) - R_2(u_2) \leqslant M. \tag{1}$$

(The latter inequality follows from the definition of M.) Similarly, $R_2(w_2) \geqslant R_2(w_1)$, so

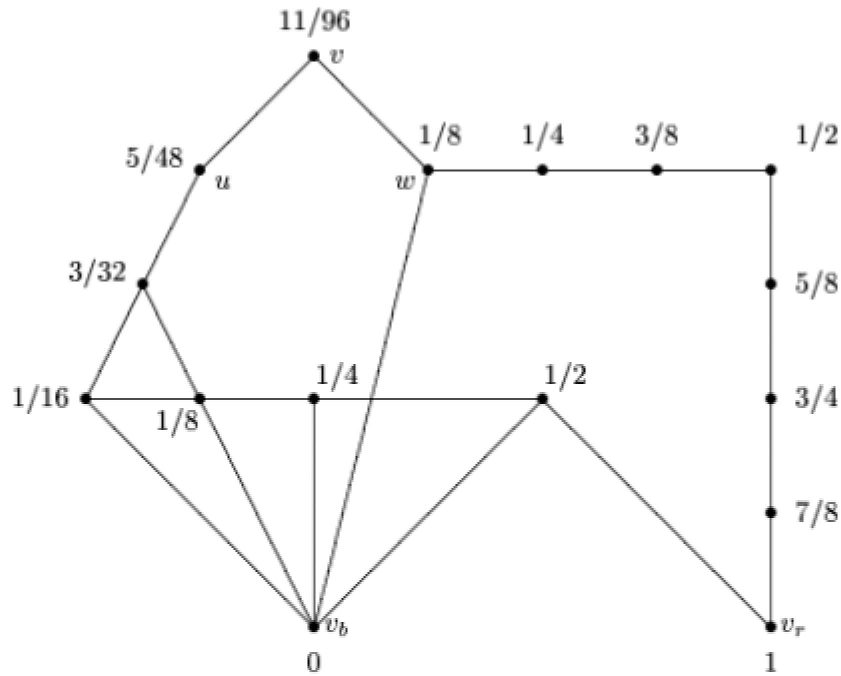
$$R_1(w_1) - R_2(w_2) \leqslant R_1(w_1) - R_2(w_1) \leqslant M. \tag{2}$$

Adding (1) and (2), we have

$$(R_1(u_1) + R_1(w_1)) - (R_2(u_2) + R_2(w_2)) \leqslant 2M. \tag{3}$$

The left side is $2R_1(v) - 2R_2(v) = 2M$, so equality must hold in (2). In particular, $R_1(u_2) - R_2(u_2) = M$; i.e., u_2 satisfies the hypothesis on v. Since u_2 was any vertex with $R_2(u_2) = R_2^{-}(v)$, induction shows that $R_1(u) - R_2(u) = M$ for all $u \in \bar{v}$, where descent is measured with respect to R_2. Since $R_1(b) - R_2(b) = 0$ and $b \in \bar{v}$, we have $R_1(v) - R_2(v) \leqslant 0$ everywhere. That is, $R_1 \leqslant R_2$. The same argument for $R_2 - R_1$ shows the opposite inequality, so $R_1 = R_2$. ∎

2.3. **Characteristic of the Optimal Strategy in Richman Games.** Lets consider the directed graph in the figure below whose Richman Cost Function is as indicated.



If the token is located at the vertex v, then Blue can win with a fraction of the money supply greater than $\frac{11}{96}$. Note that his optimal move is to the vertex u, which is simultaneously farther from this goal v_b and closer to his opponent's goal v_r. His alternative is to move to the vertex w, which is closer to his goal and farther from his opponent's. This illustrates that the optimal strategy in Richman games does not respect the usual distance function on graphs.

3. Incomplete Knowledge

It is possible to implement a winning strategy without knowing how much money the opponent has. Let's define safety ratio from the perspective of Blue.

Definition 7. Blue's safety ratio at v is the fraction of the total money that he has in his possession, divided by $R(v)$ (the fraction that he needs in order to win).

Note: Blue will not know the value of his safety ratio, since we are assuming that he has no idea how much money Red has.

Theorem 8. *Suppose Blue has a safety ratio strictly greater than 1. Then Blue has a strategy that wins with probability 1 and does not require the knowledge of Red's money supply.*

Proof. Here is Blue's strategy: When the token is at vertex v, and he has B dollars, he should act as if his safety ratio is 1; that is, he should play as if Red has R_{crit} dollars with $\frac{B}{B+R_{crit}} = R(v)$ and the total amount of money is $B + R_{crit} = \frac{B}{R(v)}$ dollars. He should accordingly bid

$$X = \frac{R(v) - R^{-}(v)}{R(v)} B$$

dollars. Suppose blue wins(by outbidding or by tiebreak) and moves to u along an edge of steepest descent. Then Blue's safety ratio changes

$$from \quad \frac{(\frac{B}{B+R})}{R(v)} \quad to \quad \frac{(\frac{B-X}{B+R})}{R(u)},$$

where R is the actual amount of money that Red has. However, these two safety ratios are actually equal, since

$$\frac{B-X}{B} = 1 - \frac{X}{B} = 1 - \frac{R(v)-R(u)}{R(v)} = \frac{R(u)}{R(v)}.$$

Now suppose instead that Red wins the bid (by outbidding or by tiebreak) and moves to z. Then Blue's safety ratio changes

$$from \quad \frac{(\frac{B}{B+R})}{R(v)} \quad to \quad \frac{(\frac{B+Y}{B+R})}{R(z)},$$

with $Y \geqslant X$. Note that the new safety ratio is greater than or equal to

$$\frac{(\frac{B+X}{B+R})}{R(w)},$$

where $R(w) = R^{+}(v)$. But this lower bound on the new safety ratio is equal to the old safety ratio, since

$$\frac{B+X}{B} = 1 + \frac{X}{B} = 1 + \frac{R(w)-R(v)}{R(v)} = \frac{R(w)}{R(v)}.$$

In either case, the safety ratio is nondecreasing, and in particular must stay greater than 1. On the other hand, if Blue were to eventually lose the game, his safety ratio at that moment would have to be at most 1, since his fraction of the total money supply cannot be greater than $R(r) = 1$. Consequently, our assumption that Blue's safety ratio started out being greater than 1 implies that Blue can never lose. In an acyclic graph, infinite play is impossible, so the game must terminate at b with a victory for Blue. ■

4. Poorman Variant

The Poorman Game has a setup very similar to that of the Richman Game. However, the two games differ in the fact that in a Poorman game, the higher bidder pays the amount of the bid to the bank instead of the lower bidder, so that the money would never be seen again. The winning strategy in the Poorman Game is governed by a different sort of cost function called the Poorman Cost Function.

Definition 9. Given $0 \leqslant x \leqslant y \leqslant 1$, we define the Poorman's average of x and y to be

$$avg_P(x,y) = \frac{y}{1-x+y}.$$

Note: $avg_P(x,y) \leqslant y$ since $1-x+y \geqslant 1$. Also, $x - avg_P(x,y) = (y-x)(1-x) \geqslant 0$, so $avg_P(x,y) \geqslant x$.

Definition 10. Given a directed graph D with distinguished vertices b and r, a Poorman Cost Function is a function $P : V(D) \to [0,1]$ such that

$$P(b) = 0, \qquad P(r) = 1,$$

and
$$P(v) = avg_P(P^-(v), P^+(v))$$
for v black.

4.1. **Theorems:**

Theorem 11. *There exists a Poorman Cost Function $P(v)$ for the directed graph D.*

Proof. Consider the auxiliary functions $p(v,t)$ and $P(v,t)$. Let
$$p(b,t) = 0, \qquad P(b,t) = 0$$
$$p(r,t) = 1, \qquad P(r,t) = 1$$
for all $t \in \mathbb{N}$. For v black, let
$$P(v,t) = avg_P(P^-(v,t-1), P^+(v,t-1)),$$
and
$$p(v,t) = avg_P(p^-(v,t-1), p^+(v,t-1)).$$
A simple induction shows that $P(v,t+1) \leqslant P(v,t)$ for all v and all $t \geqslant 0$. Therefore, $P(v,t)$ is weakly decreasing and bounded below by zero as $t \to \infty$, hence convergent. It is also evident that the function $P(v) = \lim_{t\to\infty} P(v,t)$ satisfies the definition of a Poorman Cost Function.
Similarly, $p(v,t)$ converges to a Poorman Cost Function $p(v)$. ■

Theorem 12. *Suppose Blue and Red play the Poorman Game on the directed graph D with the token initially located at vertex v. If Blue's share of the total money supply exceeds $P(v) = \lim_{t\to\infty} P(v,t)$, then he has a winning strategy. Moreover, his victory requires at most t moves if his share of the money supply exceeds $P(v,t)$.*

Proof. It suffices to prove the result concerning $P(v,t)$. Suppose it is true for $t-1$, and let Blue have x dollars and Red have y dollars where $\frac{x}{x+y} > P(v)$. Blue bids Δ dollars where
$$\Delta = \frac{(P(v,t) - P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))} = \frac{(P^+(v,t-1) - P(v,t))x}{P(v,t)P^+(v,t-1)}.$$
If a bid by Blue prevails, then his share of the money supply decreases but remains at least
$$\begin{aligned}\frac{x-\Delta}{x+y-\Delta} &= \frac{x - \frac{(P(v,t)-P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))}}{x+y-\frac{(P(v,t)-P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))}} \\ &> \frac{x - \frac{(P(v,t)-P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))}}{\frac{x}{P(v,t)} - \frac{(P(v,t)-P^-(v,t-1))x}{P(v,t)(1-P^-(v,t-1))}} \\ &= P^+(v,t-1).\end{aligned}$$
By the induction hypothesis, Blue can win in either case. ■