Cop-Win Graphs in Cops and Robbers

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August 2021

1 Introduction

We will analyze the game of Cops and Robbers. The game is played on an undirected graph by two players, the cop and the robber, who alternate turns. On each player's turn, he can move to an adjacent node of the graph, or can choose not to move. (Because of the fact that they can choose not to move, we can consider the graph to be *reflexive*; i.e. every node is connected to itself). The cop wins if at any point the cop and the robber are on the same node, and the robber is trying to prevent the cop from winning.

In the version of the game we consider, there is only one cop, and the game starts with the robber and the cop choosing their starting nodes, with the cop choosing first, and then moving first once the game has begun.

We present the proof of a theorem (Theorem 1) originally proved by R. Nowakowski and P. Winkler in their paper *Vertex-to-Vertex Pursuit in a Graph* [1], showing exactly which graphs are a win for the cop and explaining the winning strategy, as well as a proof of a related result (Theorem 2) that was mentioned in that paper, but not fully proved.

2 The Cop-Win Graphs

In order to identify which graphs are a win for the cop, we first define the following binary operation(s):

Definition 1. *Consider a class of binary relations $\leqslant_a$, on the set of nodes in a graph, defined for each ordinal a. For two nodes x and y in a graph, we say that $x \leqslant_a y$ if and only if for every node u adjacent to x there exists a node v adjacent to y and an ordinal $p < a$ such that $u \leqslant_p v$. Additionally, for the base case $a = 0$, we define $x \leqslant_0 y$ to be true if and only if $x = y$.*

Let a' be the smallest ordinal such that the relation $\leqslant_{a'}$ is equivalent to the relation $\leqslant_{a'+1}$. Let us call this particular relation $\leqslant$.

Note that in order for this relation to be well-defined, we must prove the following:

Lemma 1. *The ordinal a' exists.*

Proof. We do this by showing that, if $a < b$,

$$\{(x, y) \mid x \leqslant_a y\} \subseteq \{(x, y) \mid x \leqslant_b y\}$$

or, informally, for larger ordinals, the binary relation we defined is more permissive.

This is equivalent to the statement that, if $a < b$, then $x \leqslant_a y \implies x \leqslant_b y$. So suppose that $x \leqslant_a y$. Using the definition of $\leqslant_a$, this means that for every node u adjacent to x there is a node v adjacent to y such that $u \leqslant_c v$ for some $c < a$. But since a is less than b, so is c. So the ordinal c satisfies the criteria in the definition, and $x \leqslant_b y$.

This proves that a' exists because, as we increase a, there is now at least one more pair (x, y) where $x \leqslant_a y$. (If this was not the case, then we would have already found a'). Therefore we will eventually reach some a' where $x \leqslant_{a'} y$ for every pair of nodes x, y, and since the set of nodes (x, y) where $x \leqslant_{a'+1} y$ is a superset of the set of nodes (x, y) where $x \leqslant_{a'} y$, we must have that $\leqslant_{a'} = \leqslant_{a'+1}$. □

Now that we have shown that the operation $\leqslant$ is well-defined, we state the following theorem:

Theorem 1. *The cop has a winning strategy on a graph G if and only if $x \leqslant y$ for every node x and y in G, where $\leqslant$ is the binary operation from Definition 1.*

For example, consider a cyclic graph with four nodes. This is clearly a win for the robber; let us use Theorem 1 to verify that. If we construct the table of values for $\leqslant_0$ and $\leqslant_1$, we get Table 1. Observe that the results for $\leqslant_0$ and $\leqslant_1$ are the same, so for this example, $a' = 0$. If we look at the table of values for $\leqslant_0$, we see that it is not the case that $x \leqslant_0 y$ for every x and y. Therefore Theorem 1 would predict that this graph is a win for the robber, which is correct.

As another example, consider the complete graph with four nodes. This graph is a win for the cop: wherever the robber goes, the cop can follow in a single move, since every pair of nodes is connected. In order to use the theorem to verify this result, we construct the tables in Table 2. Observe that the table we made for $\leqslant_1$ is the same as the one for $\leqslant_2$, so $a' = 1$. Since $\leqslant_1$ includes every pair of nodes, Theorem 1 would predict that this graph is a win for the cop; this is the correct answer.

$\leqslant_0$	A	B	C	D
A	✓			
B		✓		
C			✓	
D				✓

$\leqslant_1$	A	B	C	D
A	✓			
B		✓		
C			✓	
D				✓

Table 1: Robber-Win Example. Values of $\leqslant_n$ for a cyclic graph with four nodes A, B, C, D and four edges (A, B), (B, C), (C, D), (D, A).

$\leqslant_0$	A	B	C	D
A	✓			
B		✓		
C			✓	
D				✓

$\leqslant_1$	A	B	C	D
A	✓	✓	✓	✓
B	✓	✓	✓	✓
C	✓	✓	✓	✓
D	✓	✓	✓	✓

$\leqslant_2$	A	B	C	D
A	✓	✓	✓	✓
B	✓	✓	✓	✓
C	✓	✓	✓	✓
D	✓	✓	✓	✓

Table 2: Cop-Win Example. Values of $\leqslant_n$ for a complete graph (i.e. every pair of nodes is connected) with four nodes A, B, C, D.

Proof of Theorem 1. The cop and robber choose their starting nodes x_0 and y_0 respectively. We make no assumptions about where they start; this strategy works for all starting positions. Since $y_0 \leqslant x_0$, there must be some x_1 adjacent to x_0 and some ordinal $a_0 < a'$ such that $y_0 \leqslant_{a_0} x_1$ (this is basically just a restatement of the definition of $\leqslant$). The cop's strategy begins by moving to x_1.

On subsequent turns, the cop continues this strategy: on his nth turn, if the cop is at x_n and the robber is at y_n with $x_n \leqslant_{a_n} y_n$ for some ordinal a_n, the cop moves to a vertex x_{n+1} adjacent to x such that $x_{n+1} \leqslant_{a_{n+1}} y_n$ for some $a_{n+1} < a_n$. (The fact that such a vertex will always exist follows from the definition of $\leqslant$.)

Using this strategy, the cop will eventually catch the robber: the sequence of ordinals a_n is strictly decreasing, so eventually it will reach zero. Additionally, at all times, $x_n \leqslant_{a_n} y_n$, so when $a_n = 0$, $x_n = y_n$: the cop and robber are at the same vertices, and the cop wins.

We have shown that this condition is sufficient for the cop to win: now we will show that it is necessary.

Suppose that the cop has a winning strategy starting at some node x_0. We claim that, for every node v_0, the cop also has a winning strategy starting from v_0. Because he wins starting from x_0, every game state where the cop is at x_0 is a winning position for the cop. Therefore, if forced to start at another node v_0, he could simply travel to x_0 and then follow his winning strategy. This lets us simplify the proof: if we can prove that he cannot win when forced to start at some node v_0, then this also proves that he cannot win when he is allowed to choose the starting node.

Since we are now considering the case where the statement $\forall x, y : x \leqslant y$ is false, we can say there are some nodes x_0, y_0 where $x_0 \nleqslant y_0$. Then we can force the cop to start at x_0 (see the paragraph above). Since $y_0 \nleqslant_{a'+1} x_0$, and since $\leqslant_{a'+1} = \leqslant_{a'}$, there must exist a node y_1 adjacent to y_0 such that, for every x adjacent to x_0, $y_1 \nleqslant x$. The robber starts at y_1. On the robber's nth turn, he is at some node y_n and the cop is at some node x_n, with $y_n \nleqslant x_n$. (We must have that $y_n \nleqslant x_n$ because it is true on the first turn, and if it is true on turn n, then the robber moves such that it is true on turn $n+1$). The robber chooses a node y_{n+1} such that $y_{n+1} \nleqslant x_n$. This node must exist on the nth turn for the same reason it exists on the first turn. Now, note that if, for some turn number n, $x_n = y_n$ (i.e. the robber has been caught) that would mean that $y_n \leqslant_0 x_n$, by definition, and so then $y_n \leqslant_{a'} x_n$. But the robber's strategy described above guarantees that, for every turn number n, $y_n \not\leqslant x_n$, so this can never occur, and the robber is never caught. □

Here is an interesting result that follows from this theorem:

Theorem 2. *Every incomplete regular graph with at least two nodes is a win for the robber.*

Proof. First we will prove that, for a regular graph, a' cannot be greater than 1. Observe that, using the definition of $\leqslant_n$, we can rephrase the statement $x \leqslant_1 y$ to the statement that every neighbor of x is also a neighbor of y, or that $N(x) \subseteq N(y)$, where $N(x)$ represents the set of neighbors (the "neighborhood") of x. Then we can rephrase the statement $x \leqslant_2 y$ to the statement that, for every node u adjacent to x, there is a v adjacent to y such that $N(u) \subseteq N(v)$. In order to prove that $a' \leq 1$ for every regular graph, we just need to show that $\leq_1$ is equivalent to $\leq_2$. Using a fact from the proof of Lemma 1, we reduce this statement to the statement that if $x \leq_2 y$, then $x \leq_1 y$. So, suppose that $x \leq_2 y$, i.e. every node u adjacent to x has a node v adjacent to y such that $N(u) \subseteq N(v)$. We'd like to show that $u \in N(x) \implies u \in N(y)$. Again using the definition of $\leqslant$, there must exist a v adjacent to y such that $N(u) \subseteq N(v)$. But since the graph is regular, $N(u) \subseteq N(v)$ is equivalent to $N(u) = N(v)$. Since v is adjacent to y, and $N(u) = N(v)$, we can say that u is adjacent to y. So, since $u \in N(x) \implies u \in N(y)$, we can say that $\leqslant_1$ is equivalent to $\leqslant_2$, and so $a' \leq 1$.

Now that we know that $a' \leq 1$, we'd like to prove that there is some x, y such that $x \not\leqslant_{a'} y$. First, observe that if $a' = 0$, then $x \leqslant y$ is equivalent to $x = y$ by definition, and so it is trivial to find some pair of nodes (x, y) where $x \neq y$. So we only need to prove it for the case where $a' = 1$. Since the graph is regular, $N(x) \subseteq N(y)$ is equivalent to $N(x) = N(y)$. Now we need to find some pair of nodes (x, y) where $N(x) \neq N(y)$. Since the graph is reflexive, we can do this by finding two nodes x and y that are not connected to each other; then $N(x)$ includes x but not y, and $N(y)$ includes y but not x, so $N(x) \neq N(y)$. Since the graph is incomplete, there must be at least one pair of non-adjacent nodes, so the nodes x and y must exist. Then Theorem 1 guarantees that the robber can win. □

References

[1] R. Nowakowski and P. Winkler, "Vertex-to-vertex pursuit in a graph," 1981.