Impartial Games

Euler Circle - Final Project

Rohan Das

August 14, 2021

§1 Introduction

Impartial games are combinatorial games in which both players have the same moves. The prototypical impartial game is the game of NIM. Here, there are several piles. A move consists of removing any positive amount of stones from a pile, and a player wins by removing the last stone.

In this paper, we will discuss two variations of NIM – Fibonacci NIM and Wythoff. We will go over the winning positions and strategies. To do this analysis, we need a few concepts.

Single pile NIM games are characterized by *nimbers*.

Definition 1.1 (Nimbers)**.** Nimbers are defined as

$$*n = \{*0, *1, \dots, *(n-1)\}.$$

Nimbers are computed using the Minimum Excludant (*mex*). The mex of a set is the smallest nonnegative integer not in the set, so $\operatorname{mex}(0,1,2) = 3$, $\operatorname{mex}(0,2) = 1$, and $\operatorname{mex}(2) = 0$.

Theorem 1.2 (Mex rule)

$$\{*a_1, *a_2, \dots, *a_k\} = *m,$$

where $m = \operatorname{mex}(a_1, a_2, \dots, a_k)$.

Sprague-Grundy Theory says that every impartial game is equivalent to a single pile NIM game.

Definition 1.3 (Grundy Value)**.** For an impartial game G with $G = *n$, we call n the Grundy value of G, and we write $\mathcal{G}(G) = n$.

From the Mex rule theorem and the definition of Grundy value,

$$\mathcal{G}(n) = \operatorname{mex}(\mathcal{G}(n-1), \mathcal{G}(n-2), \dots, \mathcal{G}(1), \mathcal{G}(0)).$$

§2 Fibonacci NIM

§2.1 Introduction

Fibonacci NIM is played with a pile of n stones with the following rules:

- The first move can remove up to $n-1$ stones.
- If the previous move removed k stones, the current move can remove up to $2k$ stones.
- The last player to remove a stone wins.

Note. We use the notation (n, r) when there are n stones in the pile and at most r can be removed. Note that the starting position is $(n, n-1)$, which we can abbreviate as (n).

Winning example Here, we play through the game starting with 10 stones, and left moving first. If left plays optimally, she is guarunteed to win. The sequence of moves is shown in Figure 1. The left player always has an optimal move in this case, so other moves aren't shown.

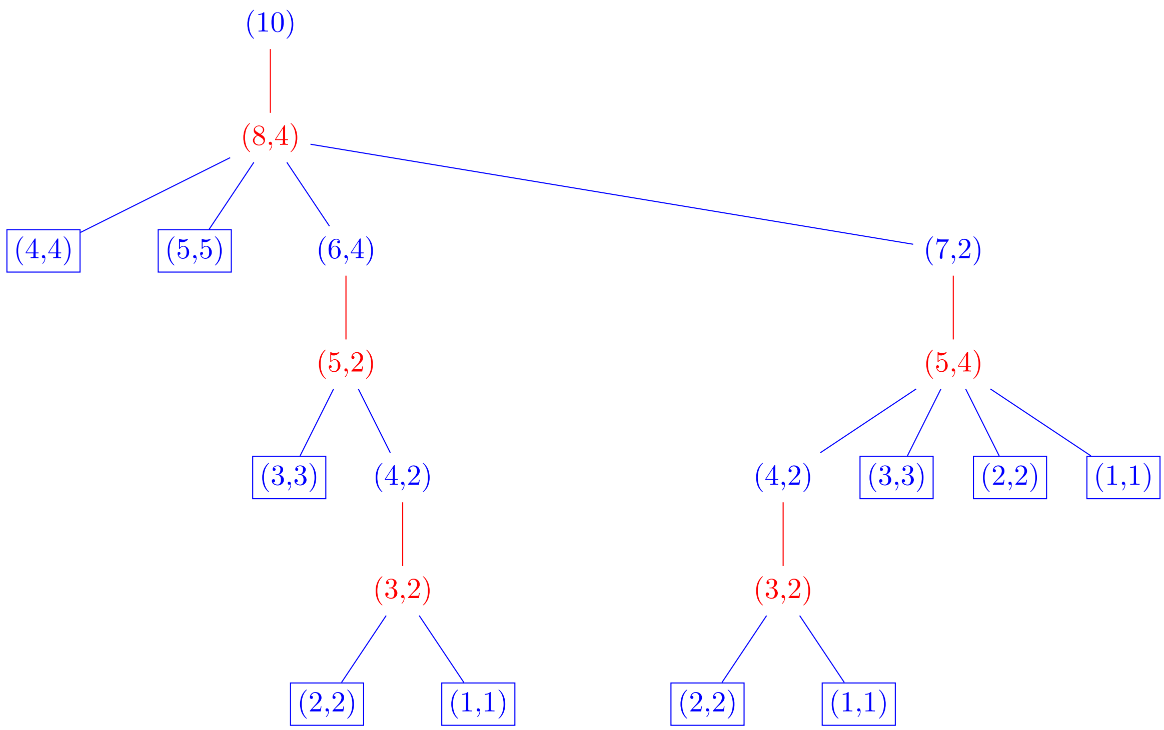

Figure 1: Fibonnaci NIM Example 1

Losing example Here, we play through the game starting with 8 stones, and left moving first. If right plays optimally, he is guarunteed to win. The sequence of moves is shown in Figure 2. The right player always has an optimal move in this case, so other moves aren't shown.

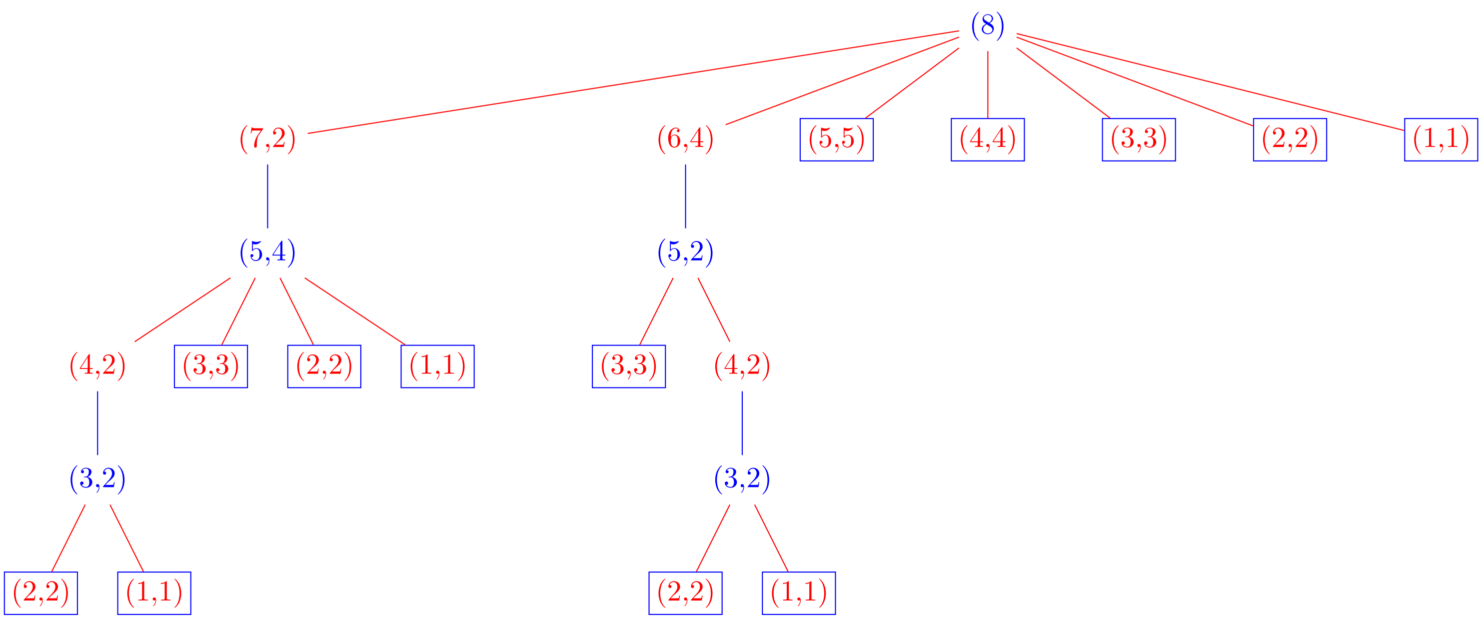

Figure 2: Fibonnaci NIM Example 2

In each of these examples, we can see that the player with a winning strategy moves to $8, 5$, and 3. These are all Fibonacci numbers, motivating the following theorems.

§2.2 Zeckendorf's Representation

The winning strategy and analysis for Fibonacci NIM depends on the Zeckendorf's represetation. We briefly review it here.

Theorem 2.1

Zeckendorf's Theorem Let n be a positive integer. Then there is a unique increasing sequence $(c_i)_{i=0}^k$ such that $c_i \geq 2$ and $c_{i+1} > c_i + 1$ for $i \geq 0$, and that

$$n = \sum_{i=0}^{k} F_{c_i}.$$

The proof for this can be found in [5].

The Zeckendorf's representation for a few numbers are shown below:

- $3 = F_4$.
- $6 = 5 + 1 = F_5 + F_2$.
- $8 = F_6$.
- $10 = 8 + 2 = F_6 + F_3$.

- $12 = 8 + 3 + 1 = F_6 + F_4 + F_2$.

Corollary 2.2

If $F_{k+1} > n \geq F_k$, then F_k is the largest number in the Zeckendorf representation of n.

Proof. By definition, for all $m > k$, $F_m > n$. Therefore, nothing larger than F_k can be in the Zeckendorf representation of n. Now, we want to show that F_k has to be in the representation.

For the sake of contradiction, let F_k not be in the representation. Then, the maximum possible value of the representation is $n' = F_{k-1} + F_{k-3} + F_{k-5} + \ldots + F_{k'}$, where k' is 2 if k is odd, and 3 if k is even. Now,

$$\begin{aligned} n' + F_{k'-1} &= F_{k-1} + F_{k-3} + F_{k-5} + \ldots + F_{k'+6} + F_{k'+4} + F_{k'+2} + F_{k'} + F_{k'-1}, \\ &= F_{k-1} + F_{k-3} + F_{k-5} + \ldots + F_{k'+6} + F_{k'+4} + F_{k'+2} + F_{k'+1}, \\ &= F_{k-1} + F_{k-3} + F_{k-5} + \ldots + F_{k'+6} + F_{k'+4} + F_{k'+3}, \\ &= F_{k-1} + F_{k-3} + F_{k-5} + \ldots + F_{k'+6} + F_{k'+5}, \\ &\ldots, \\ &= F_k. \end{aligned}$$

Therefore, $n' < F_k < n$. This is a contradiction, and therefore, the Zeckendorf representation of n must contain F_k. □

Definition 2.3. Let $n = F_{i_r} + Fi_{r-1} + \cdots + F_{i_1}$ where $r, i, n \in \mathbb{N}$. We define the *tail*, $T(n) = F_{i_1}$.

That is, $T(n)$ is the smallest number in the Zeckendorff representation.

From the earlier Zeckendorf representations, $T(3) = F_4 = 3$, $T(6) = F_2 = 1$, $T(8) = F_6 = 8$, $T(10) = F_3 = 2$, and $T(12) = F_2 = 1$.

§2.3 Winning Strategy

The Zeckendorf Representation tells us if a game is winning or losing for left (assuming left plays first). It also tells us the winning strategy for whichever player wins (assuming optimal play). Assuming that there are n stones left, if it is allowed to remove $T(n)$ stones, then the next player to play wins (such positions are described as $\mathscr{N}$). Otherwise, the next player to move loses, so the previous player wins (such positions are described as $\mathscr{P}$).

This means that if the starting position n is a Fibonacci number, the first player to move loses, because she can only remove up to $n - 1$ stones, and $T(n) = n$.

If the starting position n is not a Fibonacci number, the first player to move wins by repeatedly removing $T(n)$. We will see and prove that this is possible in the following theorem.

Theorem 2.4

Initial game is a $\mathcal{P}$ position if and only if the number of stones is a Fibonacci number.

Lemma 2.5

For every $i \in N$ where $i \geq 3$, $2F_{i-1} \geq F_i$ and $F_{i+1} > 2F_{i-1}$.

Proof. First, note that $F_{i+1} = F_i + F_{i-1} = F_{i-1} + F_{i-2} + F_{i-1} > 2F_{i-1}$ (when $F_{i-2} > 0$, so $i \geq 3$).

Now, we want to show that $2F_{i-1} \geq F_i$ for $i \geq 3$. We can show this by induction:

For the two base cases, $2F_2 = 2 \geq 2 = F_3$, and $2F_3 = 4 \geq 3 = F_4$.

Now, assume that $2F_{i-3} \geq F_{i-2}$ and $2F_{i-2} \geq F_{i-1}$. Now,

$$2F_{i-1} = 2(F_{i-2} + F_{i-3}) \geq F_{i-1} + F_{i-2} = F_i.$$

□

Claim 2.6 — $\mathscr{P}$ is the set of positions (n, r), such that $T(n) > r$. $\mathscr{N}$ is the set of positions not in $\mathscr{P}$, that is $T(n) \leq r$.

The next Lemma, Lemma 2.7, says that every move from $\mathscr{P}$ is to $\mathscr{N}$.

Lemma 2.7

Let $n \in \mathbb{N}$. For any p with $T(n) > p$, $(n-p, 2p) \in \mathscr{N}$.

Proof. For the sake of contradiction, let $(n-p, 2p) \in \mathscr{P}$. Then $T(n-p) > 2p$.

Let the Zeckendorf's Representation of $n-p$ be $F_{i_1} + F_{i_2} + \ldots$, where $F_{i_x} < F_{i_y}$ if $x < y$. Now, $T(n-p) = F_{i_1} > 2p$. By Lemma 2.6, $2F_{i_1-1} \geq 2F_{i_1} > 2p$, $p < 2F_{i_1-1}$. Therefore, the Zeckendorf's Representation of p does not contain any $F_{i'}$ with $i' \geq i_1 - 1$.

Now, because the Zeckendorf's representation of $n-p$ and p are disjoint, their sum is a valid Zeckendorf's representation of n. However, the given Zeckendorf's representation of n has $T(n) > p$, while this one contains values $\leq p$. Therefore, this is a separate Zerckendorf's representation. This contradicts it's uniqueness.

Therefore, $(n-p, 2p) \in \mathscr{N}$. □

The next Lemma, Lemma 2.8, says from $\mathscr{N}$ there is a move to $\mathscr{P}$.

Lemma 2.8

Let $n \in \mathbb{N}$. Let $p := T(n)$. Then $(n-p, 2p) \in \mathscr{P}$.

Proof. Let the Zeckendorf's Representation of n be $F_{i_1} + F_{i_2} + F_{i_3} + \ldots$ for $i_1 < i_2 < i_3 < \ldots$. By definition, $p = F_{i_1}$, so the Zeckendorf's Representation of $n-p$ is $F_{i_2} + F_{i_3} + \ldots$.

By definition, $F_{i_2} \geq F_{i_1+2}$, and by Lemma 2.6, $F_{i_1+2} > 2F_{i_1} = 2p$. Therefore, $2p < F_{i_2} = T(n-p)$, so $(n-p, 2p)$ is a $\mathscr{P}$ position. □

Now, we can define Partition Theorem. It states that

Theorem 2.9 (Parition Theorem)
If $\mathscr{N}$ and $\mathscr{P}$ are disjoint sets of positions (of impartial games) such that:

- $\mathscr{N} \cup \mathscr{P}$ is the set of all positions,
- For each $\mathscr{N}$ position, there is a move to an $\mathscr{P}$ position,
- For each $\mathscr{P}$ position, all moves are to an $\mathscr{N}$ position,

Then $\mathscr{N}$ is the set of winning positions, and $\mathscr{P}$ is the set losing positions (for the next player to move).

The general idea behind the proof of the Partition Theorem is that because impartial games are dead-end, the game has to eventually end. Then, it works upwards from there.

Proof of Claim 2.6. From Lemma 2.7, we know that every move from $\mathscr{P}$ is to $\mathscr{N}$. From Lemma 2.8, we know that there is a move from $\mathscr{N}$ to $\mathscr{P}$. By the partition theorem, $\mathscr{P}$ are the $\mathcal{P}$ positions, and $\mathscr{N}$ are the $\mathcal{N}$ positions. □

Proof of Theorem 2.4. In the starting position, any amount of stones can be taken, except the whole pile itself. Therefore, the position is $(n, n-1)$. When this is a $\mathcal{P}$ position, $T(n) > n-1$. Because $T(n) \leq n$ by definition, $T(n) = n$, so n is a Fibonacci number. □

§2.4 Grundy Values

The grundy values of (n, r) is defined as the mex over all $\mathcal{G}(n', r')$ such that there is a move from (n, r) to (n', r').

Because there are no moves in $(n, 0)$, it's Grundy value is just $\text{mex}() = 0$. Also, note that (n, k) for $k > n$ is just (n, n).

$$\begin{aligned}
\mathcal{G}(1,1) &= \text{mex}(\mathcal{G}(0,0)) = \text{mex}(0) = 1, \\
\mathcal{G}(2,1) &= \text{mex}(\mathcal{G}(1,1)) = \text{mex}(1) = 0, \\
\mathcal{G}(2,2) &= \text{mex}(\mathcal{G}(0,0), \mathcal{G}(1,1)) = \text{mex}(0,1) = 2, \\
\mathcal{G}(3,1) &= \text{mex}(\mathcal{G}(2,2)) = \text{mex}(2) = 0, \\
\mathcal{G}(3,2) &= \text{mex}(\mathcal{G}(2,2), \mathcal{G}(1,1)) = \text{mex}(2,1) = 0, \\
\mathcal{G}(3,3) &= \text{mex}(\mathcal{G}(2,2), \mathcal{G}(1,1), \mathcal{G}(0,0)) = \text{mex}(0,1,2) = 3, \\
\mathcal{G}(4,1) &= \text{mex}(\mathcal{G}(3,2)) = \text{mex}(0) = 1, \\
\mathcal{G}(4,2) &= \text{mex}(\mathcal{G}(3,2), \mathcal{G}(2,2)) = \text{mex}(0,2) = 1, \\
\mathcal{G}(4,3) &= \text{mex}(\mathcal{G}(3,2), \mathcal{G}(2,2), \mathcal{G}(1,1)) = \text{mex}(0,2,1) = 3, \\
\mathcal{G}(4,4) &= \text{mex}(\mathcal{G}(3,2), \mathcal{G}(2,2), \mathcal{G}(1,1), \mathcal{G}(0,0)) = \text{mex}(0,2,1,0) = 3.
\end{aligned}$$

This continues similarly. We can compute a larger example: $(15, 7)$

$$\begin{aligned}
\mathcal{G}(15,7) &= \text{mex}(\mathcal{G}(14,2), \mathcal{G}(13,4), \mathcal{G}(12,6), \mathcal{G}(11,8), \mathcal{G}(10,10), \mathcal{G}(9,9), \mathcal{G}(8,8)), \\
&= \text{mex}(1,0,3,5,5,5,5), \\
&= 2.
\end{aligned}$$

More Grundy values are shown in table 1.

n \ r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0																				
1	0	1																			
2	0	0	2																		
3	0	0	0	3																	
4	0	1	1	3	3																
5	0	0	0	0	0	4															
6	0	1	1	1	1	4	4														
7	0	0	2	2	2	4	4	4													
8	0	0	0	0	0	0	0	0	5												
9	0	1	1	1	1	1	1	1	5	5											
10	0	0	2	2	2	2	2	2	5	5	5										
11	0	0	0	3	3	3	3	5	5	5	5	5									
12	0	1	1	3	3	3	3	3	6	6	6	6	6								
13	0	0	0	0	0	0	0	0	0	0	0	0	0	6							
14	0	1	1	1	1	1	1	1	1	1	1	1	1	6	6						
15	0	0	2	2	2	2	2	2	2	2	2	2	2	6	6	6					
16	0	0	0	3	3	3	3	3	3	3	3	3	7	7	7	7	7				
17	0	1	1	3	3	3	3	3	3	3	3	3	3	7	7	7	7	7			
18	0	0	0	0	0	4	4	4	4	4	4	7	7	7	7	7	7	7	7		
19	0	1	1	1	1	4	4	4	4	4	4	4	7	7	7	7	7	7	7	7	
20	0	0	2	2	2	4	4	4	4	4	4	4	4	7	7	7	7	7	7	7	7

Table 1: Gundy Values for Fibonacci NIM.

§2.5 Multi-pile Fibonacci NIM

There are two variations of extending Fibonacci NIM to multiple piles. In the first variation, called the *Local* variation, the rules are applied to each pile individually, so moving in one pile doesn't affect any other piles. From analysis of NIM games, we know that such games are can be analyzed using the Nim Sum $\oplus$ (binary addition without carrying)

$$\mathcal{G}(G_1 + G_2 + G_3 + \ldots + G_k) = \mathcal{G}(G_1) \oplus \mathcal{G}(G_2) \oplus \ldots \oplus \mathcal{G}(G_k).$$

In the second variation, the rule is applied globally, so if a player removes k stones from one pile, the other player can remove up to $2k$ stones from any pile. Larsson and Rubinstein-Salzedo have analyzed the global variation in [6], which is significantly more complex.

§3 WYTHOFF

§3.1 Introduction

WYTHOFF is played with two piles. The player who removes the last stone wins, and a move can be either of the following:

- Removing any amount of stones from one pile.
- Removing the same amount of stones from both piles.

This is equivalent to the queen-cornering game, which is played on a rectangular chessboard. A queen starts somewhere on the board, and the two players take turns moving it south, west, or southwest direction. The player who moves it to the southwest corner first wins.

The queen-cornering game is more intuitive than Wythoff, so we can use it to understand Wythoff.

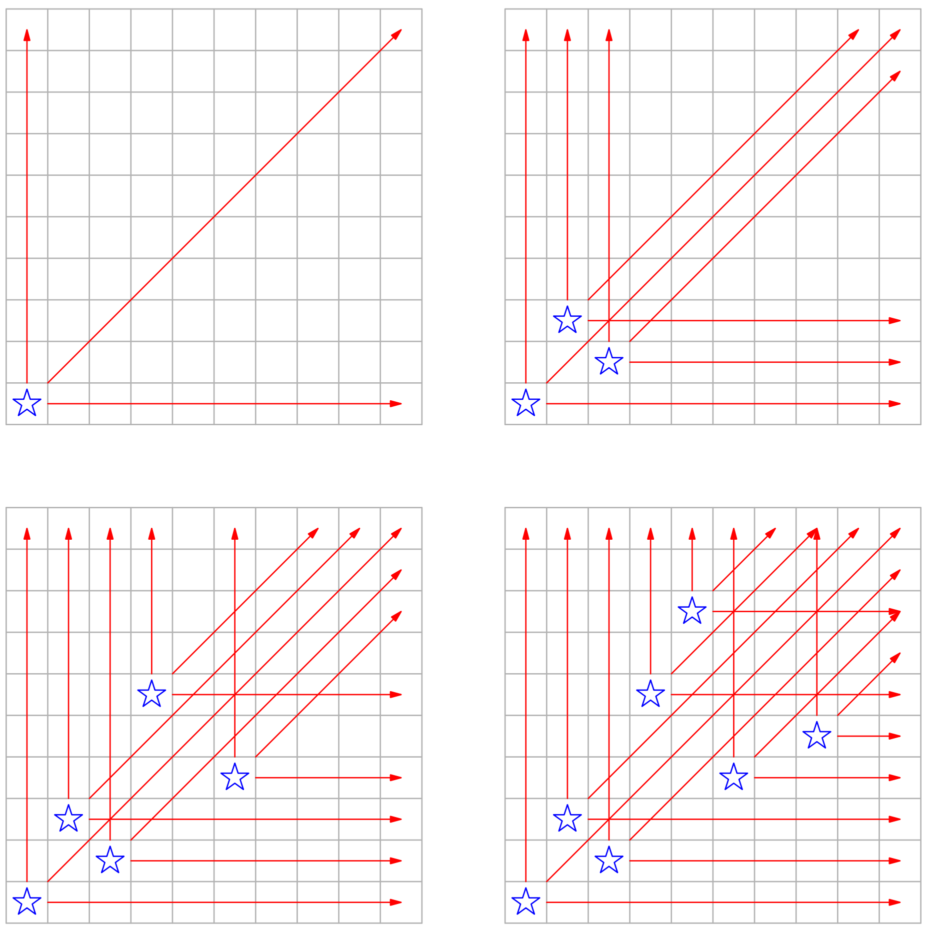

Figure 3: Queen-cornering Losing Cells

In the first diagram of figure 3, the three red rays indicate the paths to reach the southwest corner. Anything along these rays is winning for the next player to move, as she can just move to the corner instantly.

In the second diagram, the two new stars are forced to move to one of the red rays, because there is no way to get to a blue star.

Similarly, in the next diagrams, we recursively build the losing positions for the queen-cornering game on a 10×10 chessboard.

§3.2 First few Grundy values

We will start from the base case, work upwards, and then compute a larger example. The table of the first few Grundy values is shown at 2

First, note that because there are no moves from $(0,0)$, $\mathcal{G}(0,0) = \text{mex}() = 0$. Also, because order doesn't matter in Wythoff, $\mathcal{G}(a,b) = \mathcal{G}(b,a)$.

Now,

$$\mathcal{G}(n,0) = \text{mex}(\mathcal{G}(0,0), \mathcal{G}(1,0), \mathcal{G}(2,0), \ldots, \mathcal{G}(n-1,0)).$$

By induction, we can show that $\mathcal{G}(n,0) = n$.

$$\begin{aligned}
\mathcal{G}(1,1) &= \text{mex}(\mathcal{G}(0,0), \mathcal{G}(1,0), \mathcal{G}(0,1)) = \text{mex}(0,1) = 2, \\
\mathcal{G}(2,1) &= \text{mex}(\mathcal{G}(1,1), \mathcal{G}(1,0), \mathcal{G}(0,1), \mathcal{G}(2,0)) = \text{mex}(2,1,1,2) = 0, \\
\mathcal{G}(2,2) &= \text{mex}(\mathcal{G}(1,2), \mathcal{G}(0,2), \mathcal{G}(2,1), \mathcal{G}(2,0), \mathcal{G}(1,1), \mathcal{G}(0,0)) = \text{mex}(0,2,0,2,2,0) = 1, \\
\mathcal{G}(1,3) &= \text{mex}(\mathcal{G}(0,3), \mathcal{G}(1,2), \mathcal{G}(1,1), \mathcal{G}(1,0), \mathcal{G}(0,2)) = \text{mex}(3,0,2,1,2) = 4, \\
\mathcal{G}(2,3) &= \text{mex}(\mathcal{G}(1,3), \mathcal{G}(0,3), \mathcal{G}(2,2), \mathcal{G}(2,1), \mathcal{G}(2,0), \mathcal{G}(1,2), \mathcal{G}(0,1)) = \text{mex}(4,3,1,0,2,0,1) = 5, \\
\mathcal{G}(3,3) &= \text{mex}(\mathcal{G}(2,3), \mathcal{G}(1,3), \mathcal{G}(0,3), \mathcal{G}(3,2), \mathcal{G}(3,1), \mathcal{G}(3,0), \mathcal{G}(2,2), \mathcal{G}(1,1), \mathcal{G}(0,0)) \\
&= \text{mex}(5,4,3,5,4,3,1,2,0) = 6.
\end{aligned}$$

a \ b	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	0	4	5	3	7	8	6	10	11
2	2	0	1	5	3	4	8	6	7	11	9
3	3	4	5	6	2	0	1	9	10	12	8
4	4	5	3	2	7	6	9	0	1	8	13
5	5	3	4	0	6	8	10	1	2	7	12
6	6	7	8	1	9	10	3	4	5	13	0
7	7	8	6	9	0	1	4	5	3	14	15
8	8	6	7	10	1	2	5	3	4	15	16
9	9	10	11	12	8	7	13	14	15	16	17
10	10	11	9	8	13	12	0	15	16	17	14

Table 2: $\mathcal{G}$ values for Wythoff game.

§3.3 $\mathscr{P}$ positions

The $\mathscr{P}$ positions for Wythoff are given by the following lemma and theorem.

Lemma 3.1

The Wythoff $\mathscr{P}$ positions are the pairs of the form (a_n, b_n), or (b_n, a_n), where $(a_n, b_n) \in \mathbb{N}^2$ satisfy

$$\begin{aligned}
a_n &= \text{mex}\{a_i, b_i : i < n\}, \\
b_n &= a_n + n.
\end{aligned}$$

Some of the early values for A_n, B_n are shown in table 3.

n	0	1	2	3	4	5	6	7	8	9	10
A_n	0	1	3	4	6	8	9	11	12	14	16
B_n	0	2	5	7	10	13	15	18	20	23	26

Table 3: $\mathscr{P}$ positions for WYTHOFF.

The above lemma can be used to recursively compute the $\mathscr{P}$ positions. The python code shown in 1 uses this lemma to generate these $\mathscr{P}$ positions.

```python

def mex(mex_set):
    ret = 0
    while ret in mex_set:
        ret += 1
    return ret

def wyth_p_positions(max_n = 10):
    mex_set = set()
    ret = []
    for n in range(max_n+1):
        a_n = mex(mex_set)
        b_n = a_n + n

        mex_set.add(a_n)
        mex_set.add(b_n)

        ret.append((a_n, b_n))
        ret.append((b_n, a_n))

    return ret
```

Listing 1: Python to Generate $\mathscr{P}$ Positions.

The values computed by this code are plotted in Figure 4.

Note. The $\mathscr{P}$ values are close to the lines ϕx and $\frac{1}{\phi}x$.

Proof. Proof of Lemma 3.1 We can prove this by the Partition Theorem.

First, we want to show that there is no move from (a_n, b_n) to (a_m, b_m). Because the pile size can't increase, $m < n$. Due to the mex definition of a_n, $a_n \neq a_m$.

Now, since the values in the mex of a_n contains all the values in the mex of a_m, we can expand this to $a_n > a_m$. It follows that $b_n = a_n + n > a_n + m > a_m + m = b_m$, so $b_n \neq b_m$.

Additionally, $b_n - b_m = (a_n + n) - (a_m + m) = a_n - a_m + (n - m) \neq a_n - a_m$. In conclusion, there is no move from (a_n, b_n) to (a_m, b_m).

Now, we want to show that if $(a, b) \neq (a_n, b_n)$, then there is a move to some (a_n, b_n).

Without loss of generality, let $a \leq b$. Due to the mex definition, a can be expressed as either a_n or b_n.

Case 1: a can be expressed as a_n. Now, we know $b \neq b_n$ by definition. If $b > b_n$, then removing $b - b_n$ stones from the second pile results in (a_n, b_n).

If $b < b_n$, we know $a_n \leq b < b_n$. Now, $b - a = b - a_n < b_n - a_n = n$. Therefore, if $m = b - a$, $m < n$. Now, $a_n - a_m = (a_n + m) - (a_m + m) = b - b_m$. Therefore, removing stones from both piles results in (a_m, b_m).

Case 2: a can be expressed as b_n. Since $b \geq a > a_n$, removing stones from the second pile results in (b_n, a_n). □

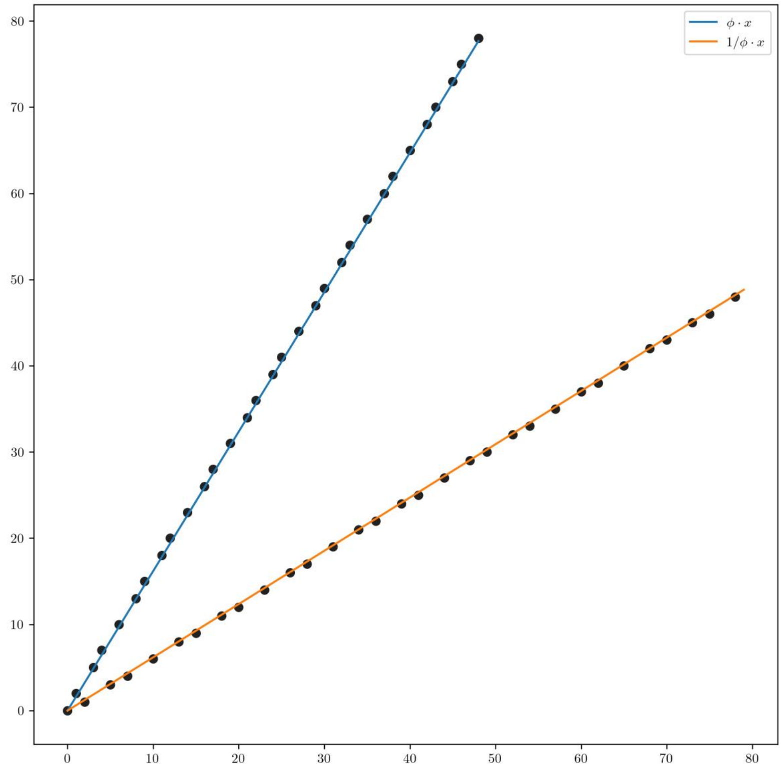

Figure 4: $\mathscr{P}$ positions.

Definition 3.2. Let $\mathcal{A}, \mathcal{B} \subset \mathbb{N}^+$. We say $\mathcal{A}$ and $\mathcal{B}$ are complementary if $\mathcal{B} = \mathbb{N}^+ \setminus \mathcal{A}$, that is, if $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $\mathcal{A} \cup \mathcal{B} = \mathbb{N}^+$.

Lemma 3.3

Let $\alpha, \beta > 1$ be irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Then

$$\{\lfloor i\alpha \rfloor, i \in \mathbb{N}^+\} \text{ and } \{\lfloor j\beta \rfloor, j \in \mathbb{N}^+\}$$

are complementary.

Proof. We can define $\mathcal{A}, \mathcal{B} \subset \mathbb{R}$ as $\mathcal{A} = \{\frac{i}{\alpha}, i \in \mathbb{N}^+\}$ and $\mathcal{B} = \{\frac{j}{\beta}, j \in \mathbb{N}^+\}$.

First, we claim that these are disjoint. This is because, when we multiply both sides of $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ by α, we get $\frac{\alpha}{\beta} = \alpha - 1$, which is irrational. For the sake of contradiction, assume that for positive integers i, j, $\frac{i}{\alpha} = \frac{j}{\beta}$. Then, $\frac{j}{i} = \frac{\alpha}{\beta}$ is irrational, which is a contradiction.

Now, let $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. For $c \in \mathcal{C}$, we can define $f(c)$ as the number of elements $c' \in \mathcal{C}$ such that $c' \leq c$.

If $c_1 \neq c_2$, we claim that $f(c_1) \neq f(c_2)$. Without loss of generality, let $c_1 < c_2$. Now, each element counted in $f(c_1)$ is also counted in $f(c_2)$. Meanwhile, c_2 is counted in $f(c_2)$ but not $f(c_1)$. Therefore, $f(c_1) \neq f(c_2)$.

Let $\mathscr{A}$ be the set of $f(a)$ for all $a \in \mathcal{A}$. Similarly, let $\mathscr{B}$ be the set of $f(b)$ for all $b \in \mathcal{B}$. As shown above, they are disjoint. Additionally, every positive integer is $f(c)$ for some $c \in \mathcal{C}$, so $\mathscr{A} \cup \mathscr{B} = \mathbb{N}^+$.

Claim — $\mathscr{A} = \{\lfloor i\beta \rfloor, i \in \mathbb{N}^+\}$ and $\mathscr{B} = \{\lfloor j\alpha \rfloor, j \in \mathbb{N}^+\}$.

Let $a \in \mathcal{A}$. Then, we have $a = \frac{i}{\alpha}$ for some positive integer i. We can separate $f(a)$ into two parts: $f_{\mathcal{A}}(a) = |\{a' \in \mathcal{A}, a' \leq a\}|$, and $f_{\mathcal{B}} = |\{b' \in \mathcal{B}, b' \leq a\}|$.

Now, $f_{\mathcal{A}}(a) = i$, and:

$$\begin{aligned} f_{\mathcal{B}}(a) &= |\{j \in \mathbb{N}^+, \frac{j}{\beta} \leq a\}|, \\ &= |\{j \in \mathbb{N}^+, j \leq a\beta\}|, \\ &= \lfloor a\beta \rfloor. \end{aligned}$$

Because $a = \frac{i}{\alpha}$,

$$\lfloor a\beta \rfloor = \left\lfloor i\left(\frac{\beta}{\alpha}\right) \right\rfloor = \lfloor i(\beta - 1) \rfloor = \lfloor i\beta - i \rfloor = \lfloor i\beta \rfloor - i.$$

Now, $f(a) = f_{\mathcal{A}}(a) + f_{\mathcal{B}}(a) = i + \lfloor i\beta \rfloor - i = \lfloor i\beta \rfloor$. Therefore, $\mathscr{A} = \{\lfloor i\beta \rfloor, i \in \mathbb{N}^+\}$. We can use this same idea to show that $\mathscr{B} = \{\lfloor j\alpha \rfloor, j \in \mathbb{N}^+\}$.

Because $\mathscr{A} \cup \mathscr{B} = \mathbb{N}^+$, we have the desired result. □

Theorem 3.4

The n^{th} $\mathscr{P}$-position of WYTHOFF is given by

$$(a_n, b_n) = (\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)$$

Proof. The properties of ϕ state that $\phi^2 = \phi + 1$, and that ϕ is irrational. From the first property, it follows that $\frac{1}{\phi^2} + \frac{1}{\phi} = 1$. Substituting $\phi^2 = \phi + 1$, $\frac{1}{\phi+1} + \frac{1}{\phi} = 1$. By Lemma 3.3, $\lfloor n(\phi + 1) \rfloor$ and $\lfloor n\phi \rfloor$ are complementary.

Because $\lfloor n\phi \rfloor$ and $\lfloor n\phi \rfloor + n$ are increasing, and are complementary,

$$\lfloor n\phi \rfloor = \operatorname{mex}(\{\lfloor m\phi, m\phi + m; m < n\}),$$

and

$$\lfloor n\phi^2 \rfloor = \lfloor n\phi + n \rfloor = \lfloor n\phi \rfloor + n.$$

Finally, by Lemma 3.1, $(a_n, b_n) = (\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)$. □

References

[1] Aaron N. Siegel, Combinatorial Game Theory

[2] Allen, C., Ponomarenko, V. (2014), Fibonacci Nim and a full characterization of winning moves, Involve, doi:10.2140/involve.2014.7.807

[3] Beatty sequence, Wikipedia. 5 June 2021, https://en.wikipedia.org/wiki/Beatty_sequence

[4] Fibonacci nim, Wikipedia. 5 June 2021, https://en.wikipedia.org/wiki/Fibonacci_nim.

[5] Henderson, N. What is Zeckendorf's Theorem? https://math.osu.edu/sites/math.osu.edu/files/henderson_zeckendorf.pdf

[6] Larsson, U., Rubinstein-Salzedo, S. Global Fibonacci nim. Int J Game Theory 47, 595–611 (2018).

[7] Larsson, U., Rubinstein-Salzedo, S. Grundy values of Fibonacci nim. Int J Game Theory 45, 617–625 (2016). https://doi.org/10.1007/s00182-015-0473-y

[8] Rubinstein-Salzedo, S. Combinatorial Game Theory. https://doi.org/10.1007/s00182-017-0574-x