# Mancala-like Games

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**Abstract**

In this paper, we look at various sowing games and mention few introductory results about these games. First, we briefly analyze the games SOWING and ATOMIC WARI. Then, we will explore some recent findings which link the endgame positions of AYO, a Mancala game, to a Russian solitaire game called TCHOUKA with a similar setup. Finally, we explore some number-theory based theorems with regards to several varieties of solitaire sowing games.

## 1 Introduction and Sowing

This paper will resolve around the concept of a *sowing game*, defined by John Conway as follows:

**Definition 1.1.** A *sowing game* is, loosely, a game where a position is a row of pots with some number of seeds in each pot. On their turn, a player picks up all the seeds from a given pot and places ("sows") them one by one into consecutive pots. Ending conditions and specific rules differ between games.

The simplest sowing game is just the game SOWING, which was invented by John Conway as a way to get a handle on the field [2]. In the game, a position is a series of some number of pots with some numbers of seeds in each of them, and it is represented by a string of numbers giving the number of seeds in each pot. A move, then, like in most Mancala-style games, is to pick up all the seeds in one pot and *sow* them one by one into the adjacent pots. This game can be made either partisan (in which case Right can only move seeds to the left and Left can only move seeds to the right) or impartial (both players can move seeds in either direction). We use the following shorthand for notational convenience.

$$\mathbf{1}^n = \underbrace{111\cdots1}_{\mathbf{n}}$$

To start with, under the rules of the impartial version, it is worth noting that it is possible to convert a string of $n$ pots with 1 seed in each of them into one pot with $n$ seeds in it. This is achieved inductively: for the base case of $n = 2$, we can just shift the position $\mathbf{11}$ to $\mathbf{02}$ in a single move. Then, in the inductive step, we start with the string $\mathbf{1}^n$ .
By the inductive hypothesis, we know that we can group $n-1$ of the stones together to create the position $\mathbf{10^{n-2}(n-1)}$. Then, by sowing everything in the rightmost pot, we get $\mathbf{21^{n-2}0}$. Now, the inductive hypothesis tells us that we can turn $\mathbf{1}^{n-2}$ into the position $\mathbf{0}^{n-3}(n-2)$, so we can turn this into $\mathbf{20^{n-3}(n-2)}$, which goes to $\mathbf{31^{n-3}0}$. We can continue doing this recursively to finally get $\mathbf{n1^{n-n} = n}$

### 1.1 Values of some sowing positions

1. $(\mathbf{10})^m\mathbf{03}(\mathbf{01})^n = \mathbf{0}$ for all m and n

   *Proof.* If Left goes first, she loses immediately, since she has no legal moves. If Right goes first, his only legal move is to the position $(\mathbf{10})^{m-1}\mathbf{211}(\mathbf{01})^n$, from which Left can move to $(\mathbf{10})^{m-1}\mathbf{0220}(\mathbf{01})^n$, and now Right has no moves. Thus, the second player always wins. ☕

2. $\mathbf{(01)^m2(01)^n} = n+1$, for all $m$ and $n$ except $m = n = 0$

*Proof.* Right has no legal moves. If $n = 0$, Left has only one legal move, to $(\mathbf{10})^{m-1}\mathbf{03} = \mathbf{0}$ by the previous theorem. Otherwise, Left has exactly two legal moves, to $(\mathbf{01})^{m+1}\mathbf{2}(\mathbf{01})^{n-1} = n$ by induction, and to $(\mathbf{10})^{m-1}\mathbf{3}(\mathbf{01})^{n} = 0$, which is a terminal position. Thus the position is equivalent to $\{n \mid \} = n + 1$ ☕

$$\begin{aligned} \mathbf{211} &= \tfrac{1}{2} \\ \mathbf{12202} &= \tfrac{1}{4} \\ \mathbf{122011} &= \tfrac{1}{8} \\ \mathbf{2121202} &= \tfrac{1}{16} \end{aligned}$$

$$\begin{aligned} \mathbf{2202} &= \uparrow \\ \mathbf{31011} &= \uparrow * \\ \mathbf{201321} &= \Uparrow \\ \mathbf{22011} &= \Uparrow * \\ \mathbf{122112} &= \uparrow\uparrow\uparrow \end{aligned}$$

$$\begin{aligned} \bullet\mathbf{2011} &= +_{1} \\ \bullet\mathbf{013} &= +_{2} \\ \bullet\mathbf{0114} &= +_{3} \\ \mathbf{332011} &= +_{1/2} \end{aligned}$$

**Figure 1.** Some interesting partisan sowing values.

## 1.2 Open Questions

For what values of n do SOWING positions exist with values $2^{-n}$, and if they all exist, can we systematically construct them? Is there a simple algorithm that splits SOWING positions into multiple independent components? Are there any other high-level simplification rules that would allow faster evaluation?

# 2 Atomic Wari

The second sowing game we consider, is called ATOMIC WARI. It is loosely based on a different family of African games, variously called wari or oware. The board is the same as in SOWING, but it is assumed that there are infinite pots. A legal move consists of taking all the seeds from one pot, and sowing them to the left or right, starting with the original pot. As in SOWING, Left moves seeds to the right; Right moves them to the left(Here we are considering partisan ATOMIC WARI, but there is also a naturally defined impartial version of this, where Left and Right can move in either direction). To avoid trivial infinite play, it is illegal to start a move at a pot that contains only one seed. At the end of a move, if the last pot in which a seed was dropped contains either two or three seeds, those seeds are captured, that is, removed from the game. Multiple captures are possible: after any capture, if the previous pot has two or three seeds, they are also captured. The game ends when there are no more legal moves, or equivalently, when no pot contains more than one seed. The first player who is unable to move loses.

For example, consider the position **312**. Left can sow the contents of the first pot, then capture the contents of the other two pots, leaving the position $\mathbf{100} = \mathbf{1}$. Left can also sow the contents of the rightmost pot, leaving the position **3111**. Right can move to either **301** or **11112**. Thus, the ATOMIC WARI position **312** has the following value:

$$\begin{aligned} \mathbf{312} &= \{\mathbf{1}, \mathbf{3111} | \mathbf{301}, \mathbf{11112}\} \\ &= \{\mathbf{0}, \{\mathbf{1001} | \mathbf{111111}\} | \{\mathbf{11} | \mathbf{11101}\}, \{\mathbf{111111} | \mathbf{11191}\}\} \\ &= \{0, * | *, *\} \\ &= \uparrow . \end{aligned}$$

ATOMIC WARI is an all small game. This is because in the game, it is assumed there are infinite pots on either side of a given pot. So, if left has a move, there must be some seeds in at least one pot and hence right also has a move. Since it is an all-small game, all positions have infinitessimal value. In the presence of remote stars, correct play in a collection of ATOMIC WARI positions is completely determined by the position's atomic weights(Hence, the name).

Except for deleting leading and trailing empty pots, there don't seem to be any clear-cut rules for simplifying ATOMIC WARI positions. The situation is similar to Sowing. Positions can often be split into sums independent components by hand, but no algorithm is known to find such splits in general. For example, $\mathbf{1231110101311} = \mathbf{123111} + \mathbf{1311}$. Similarly, there are several cases where the first or last pot contains only one seed, where the position's value does not change when this pot is removed, but no algorithm is known for detecting such positions. For example, $\mathbf{1001321} = \mathbf{1321}$

### 2.1 Sparse Atomic Wari

We call an ATOMIC WARI position "sparse" if every pot has two or fewer seeds.

**Theorem 2.1.** *Every sparse ATOMIC WARI position has the same value as the corresponding Impartial ATOMIC WARI position, and any such position can be split into independent components by removing all pots with fewer than two seeds.*

*Proof.* We prove the claim by induction on the number of deuces (pots with two seeds). The base case, in which each pot contains either one seed or none, is trivial. Both the impartial and the partial ATOMIC WARI position will have value 0. Now consider a position $X$ with n deuces, and let $X'$ denote the sum of positions obtained by deleting pots with fewer than two seeds. Each of the options of $X$ is a sparse position with either $n-1$ or $n-2$ deuces. For each move by Left, there is a corresponding move by Right in the same contiguous "string" of deuces that results in exactly the same position, once the inductive hypothesis is applied. For example, given the position $\mathbf{1222201}$, the Left move to $\mathbf{1210201} = \mathbf{2} + \mathbf{2}$ is matched by the Right move to $\mathbf{1201201} = \mathbf{2} + \mathbf{2}$. Clearly, $X$ and $X'$ have the same options, once the inductive hypothesis is applied. The theorem follows immediately. ☕

### 2.2 Open questions

Are noninteger or nonnumeric atomic weights possible in ATOMIC WARI? How can we systematically construct Impartial ATOMIC WARI positions with value $*n$ for any $n$?

## 3 Ayo and Tchoukaillon

In this section we discuss results which have been derived relating two sowing-style games, called AYO and TCHOUKAILLON. The definitions of the games are as follows:

- In AYO, players play on a board with two rows of $n$ pits each. One side of the board is said to "belong" to each of the players (thus, it is a partisan game). Some number of game tokens (called "stones") are scattered among the pits. Then, on their turn, a player can pick up all the stones in one pit on their side of the board and sow them in a successive counterclockwise direction around the board. If the last stone to be placed causes the pits it lands on to contain two or three stones, and that pit is on the opponents side of the board, the pit is captured and the stones are removed from the game.

  For the winning condition, the first player who is unable to move *so that their opponent has a move on the next turn* loses. Note that this is an almost misére-like ending condition (after all, if a player captured all of the stones on their opponents side of the board they would lose).

- In TCHOUKAILLON, the board only contains a single row of $n$ pits, with a final, larger pit to the right of all of the others which is called the Roumba. This is a solitaire game, meaning that the only outcome classes are a game which is winnable for the person playing and a game that is not. On their turn, the player selects all the stones in one pit and sows them one by one in the direction of the Roumba. However, they are not allowed to sow stones if doing so would "overshoot" the Roumba. If the last stone lands in a non-Roumba pit, they have to sow that pit as well. They win if they can get all of the stones into the Roumba.

## 3.1 Starting results for Ayo

Because AYO is a fairly complicated game, the specific type of arrangement that we will analyze is one known as a *determined position.* A determined position is a position for which one player (who we will call $S$) is able to move on each turn such that their opponent (who we call $N$) only has one stone on their side of the board and thus only one possible move. Because of the way that seeds are sown in this game, it is clear that the one stone on $N$'s side of the board must be in the first pit on their side at the beginning of their turn. This is the case so that they will have to move that one stone to pit 2, at which point, $S$ can capture the stone in pit two and deposit one stone in pit 1 of the board as they do that. If $N$'s stone were any further along on their side of the board, $S$ would not be able to capture that stone while still leaving them only one for the next turn. Note that these games are called determined because $S$ controls all of $N$'s moves, ensuring that they will win.

*Example.* Here, we will play out the following determined AYO position ($S$'s stones are on the bottom row and $N$'s are on the top):

| | | |
|---|---|---|
| 0 | 0 | 1 |
| 4 | 2 | 0 |

On their first turn, $\mathcal{N}$ must move their one seed to the next pit. Then, $\mathcal{S}$ can sow the pit with size four to reach the following arrangement:

| | | |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 3 | 1 |

$\mathcal{N}$ must move their stone again, and now $\mathcal{S}$ can sow the pit with 3 seeds in it:

| | | |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 0 | 2 |

On their next turn, $\mathcal{N}$ will make the same move for the third time, and now $\mathcal{S}$ can win by capturing their opponent's seed with their last pit of two seeds and leaving only one seed on their opponent's side.

## 3.2 Starting results for Tchoukaillon

TCHOUKAILLON is a comparatively simpler game. To start with, in order to simplify analysis, we claim that for any positive integer $s$ there is exactly one winnable (meaning that the one player has a winning strategy) position with $s$ stones. To do this, we will first need to demonstrate an important lemma:

**Lemma 3.1.** *If there is a winning move in a specific TCHOUKAILLON position, there is only one such winning move.*

*Proof.* Say that there are two or more harvestable positions. We claim that the only good option is to harvest the smallest one among them. If we instead harvest a larger one, than the smallest one will have too many stones and thus be unharvestable. As a result, the player will lose no matter what else they do in a game (since you can never remove stones from a pit if it is too big to sow). ☕

Having shown this, we are ready to show the main result:

**Theorem 3.2.** *There is one and only one winning in position which contains $s$ stones for each positive integer $s$.*

*Proof.* We will accomplish this by induction. To start, say that it holds true for all $s$ less than or equal to some integer $k$. Now, if there were a winnable position with $k+1$ stones, the unique first move would be to sow the smallest sowable pit such that we are left with a winnable position of $k$ stones. If we start with a winnable position of $k$ stones, then, we can reverse the process of sowing: put one stone in the Roumba, and then collect stones, starting in the Roumba, from right to left. Then, once you arrive at the first empty pit, put all of the collected stones in it. It is easy to see that this undoes the unique winning move to a $k$-stone winning position and thus generates the only possible $k+1$-stone winning position. With this insight, the proof is complete. ☕

### 3.3 Unification of the two games

Now, it is time to prove the central result of this section:

**Theorem 3.3.** *Determined* AYO *positions are bijectively related to winnable* TCHOUKAILLON *positions. Indeed, we take all of the seeds in the pits of a* TCHOUKAILLON *position and move them to the winning player's row of an* AYO *position, then put a stone in pit* 2 *of the other side, and the result will be the unique determined* AYO *position with that many stones.*

*Proof.* Consider the AYO position $A$ derived from some winnable AYO position $T$. We claim that $A$ is a determined position. To prove this, note that it will be determined if and only if the first player can move on each turn such that exactly two seeds spills over onto their opponents side of the board (capturing their opponent's stone and putting down a new one). However, this exactly equivalent to the TCHOUKAILLON goal of moving on each turn so that exactly one stone falls into the Roumba (we just need to shift the stones up by one to account for the difference between putting in one stone and two). Thus, because $T$ is a determined position, the first player in AYO will be able to leave their opponent with exactly one stone in pit 1 on each turn until they run out of stones. This is exactly the definition of a determined position, and so the proof follows from here. ☕

# 4 Periodicity of Tchoukaillon

The number of stones in each pit in the winning arrangements of TCHOUKAILLON display an interesting periodicity ( [1]), which we will analyze here. This task starts by proving the following lemma:

**Lemma 4.1.** *For all positive $i$, the sum of the seeds in the first $i$ pits of a winning* TCHOUKAILLON *arrangement with $n$ stones is congruent to $n \pmod{i+1}$.*

*Remark* 4.2. Note that this allows us to recursively derive the number of seeds in each pit of the winning position with $n$ seeds by using modular arithmetic. As long as we know the number of seeds in each of the first $i-1$ pits, we can take their sum $S$, and then the number of seeds in the $i^{\text{th}}$ will be $n - S \pmod{i+1}$. The same result can be accomplished by "unplaying" a TCHOUKAILLON position by trying to reverse the unique winning move at each step, but this approach is considerably more efficient.

*Proof.* This is accomplished mainly by induction. To start with, assume that it holds for some $n$. To go from the case of $n$ seeds to the one of $n+1$ seeds, we can unplay a single turn as follows: locate the first empty pot, and place one stone from all previous pots plus only extra stone from the Roumba in that pot (note that this undoes the one winning move for the $n+1$-seed TCHOUKAILLON position. Say that this empty pot is at position $x$. If $x$ is less than or equal to $i$, then only the seeds under pit $i$ will change, and the overall number will increase by exactly one. Therefore, the sum of the seeds in pits less than or equal to $i$ will go from $n \pmod{i+1}$ to $n+1 \pmod{i+1}$, which suits our induction perfectly. On the other hand, if $x$ is greater than $i$, then all the pits under $i$ including $i$ will lose one stone. Then, the number will decrease by $i$, which increases it by one $\pmod{i+1}$, so the result is the same as before. ☕

Now, we can harness this lemma to prove an important theorem:

**Theorem 4.3.** *The contents of the first $i$ pits in the arrangement with $n$ stones are periodic with period* $\operatorname{lcm}(1, 2, 3, \ldots, i+1)$.

*Proof.* We set $m = \operatorname{lcm} 1, 2, 3, \ldots, i+1)$. To prove this result, we will need to show that for all positions less than or equal to $i$, the number of stones in pit $i$ is the same for $n$ stones as it is for $n+m$ stones. First of all, the number of stones in pit 1 will be equal in both arrangements, because $m$ is a multiple of 2 and so it will not change the congruence class of pit 1 in the equation derived above. Similarly, because $n+m \equiv n$ (mod 3), and $x$ is the number of stones in pit 1 for both arrangements, we will have that the number of stones in pit 2 for the case with $n$ stones is $n - x \pmod 3$, and the number of stones when there are $m+n$ stones is $m+n-x \pmod 3$, which are equal. The proof is identical for all of the other positions. Now, to prove that this is the minimum possible period, we just observe that if any of the integers from 1 to $i+1$ did not go into $m$, then one of the equation derived above would be untrue, and so the number of stones in that pit for $n$ and $m+n$ total stones would not be the same. Having shown this, the proof is complete. ☕

*Remark* 4.4. Here are the contents of the piles for small values of $n$, where the periodicity can already be observed:

| Stones $s$ | Pit 1 | Pit 2 | Pit 3 | Pit 4 | Pit 5 | Pit 6 | Pit 7 | Pit 8 | Harvest $h_s$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | |
| 1 | **1** | | | | | | | | 1 |
| 2 | | **2** | | | | | | | 2 |
| 3 | **1** | 2 | | | | | | | 1 |
| 4 | | 1 | **3** | | | | | | 3 |
| 5 | **1** | 1 | 3 | | | | | | 1 |
| 6 | | | 2 | **4** | | | | | 4 |
| 7 | **1** | | 2 | 4 | | | | | 1 |
| 8 | | **2** | 2 | 4 | | | | | 2 |
| 9 | **1** | 2 | 2 | 4 | | | | | 1 |
| 10 | | 1 | 1 | 3 | **5** | | | | 5 |
| 11 | **1** | 1 | 1 | 3 | 5 | | | | 1 |
| 12 | | | | 2 | 4 | **6** | | | 6 |
| 13 | **1** | | | 2 | 4 | 6 | | | 1 |
| 14 | | **2** | | 2 | 4 | 6 | | | 2 |
| 15 | **1** | 2 | | 2 | 4 | 6 | | | 1 |
| 16 | | 1 | **3** | 2 | 4 | 6 | | | 3 |
| 17 | **1** | 1 | 3 | 2 | 4 | 6 | | | 1 |
| 18 | | | 2 | 1 | 3 | 5 | **7** | | 7 |
| 19 | **1** | | 2 | 1 | 3 | 5 | 7 | | 1 |
| 20 | | **2** | 2 | 1 | 3 | 5 | 7 | | 2 |
| 21 | **1** | 2 | 2 | 1 | 3 | 5 | 7 | | 1 |
| 22 | | 1 | 1 | | 2 | 4 | 6 | **8** | 8 |
| 23 | **1** | 1 | 1 | | 2 | 4 | 6 | 8 | 1 |
| 24 | | | | **4** | 2 | 4 | 6 | 8 | 1 |

**Figure 2.** The number of stones in each pit for the first winning positions

# 5 Extensions to wrapping and chaining in Tchoukaillon

One of the most popular sowing games is MANCALA, an ancient African game. In this game, the objective is for the player to collect as many seeds in their store as possible. For our purposes, let's call the two players Left and Right. In a typical game, three to four seeds are placed in each pit, and none in the stores. Left begins the game by taking all the seeds from any pit and spreading them to the right (or looping around to the left, depending), dropping one seed per pit. Note that Left may end up dropping a seed in Right's store depending on which pit they started with. Next, Right does the same thing, and then the two alternate until no seeds remain in the pits. The player with more seeds in their store is declared the winner.

Let us now introduce TCHOUKAILLON-WITH-WRAPPING, which is a variant of the regular TCHOUKAILLON. The board in this variant is the result of connecting the ends of the Tchoukaillon board, forming a circular board. The game consists of sowing seeds counterclockwise, which demonstrates the notion of "wrapping" (i.e., sowing seeds at least one time around the board and ending in the Roumba), similarly to standard Mancala ( [3]).

*Example.* Begin with a TCHOUKAILLON-WITH-WRAPPING board $(4, 7, 0, 2)$ as in **Figure 2**. First we play bin 2, from which we obtain $(6, 1, 1, 3)$. Then, playing bin 1, we obtain $(1, 2, 2, 4)$, which is winnable because it is a non-wrapping TCHOUKAILLON board.

Next, we can look at the concept of "chaining" in games like TCHOUKAILLON. First, let's look at the game TCHUKA ROUMBA. Like TCHOUKAILLON, this game is played on a one-rank board with the rightmost pit being the Roumba. The number of seeds per pit is equivalent in the starting position. A player begins by taking all seeds from one of the pits and sowing them closer and closer to the Roumba. If the final seed in this turn lands in a non-empty pit, then the contents are sowed in another lap. If the final seed lands in the Roumba, the player takes another turn sowing. However, if the final seed lands in an empty pit, then the game is terminated. Not unlike in a standard game of MANCALA, we see the concept of "chaining" in action for TCHUKA ROUMBA–if a seed lands in a non-empty pit, the play continues to sow with both that seed and

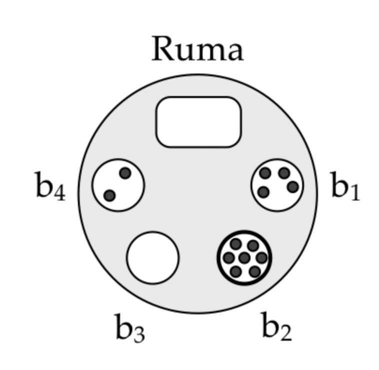


**Figure 3.** Winning Tchoukaillon-with-wrapping board.

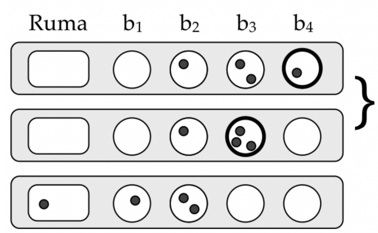


**Figure 4.** Winning Tchoukaillon-with-chaining board.

the seeds already in the pit. We can also see chaining in the game TCHOUKAILLON-WITH-CHAINING. This game uses a linear TCHOUKAILLON board, sowing towards the Roumba as usual.

*Example.* We have a winning TCHOUKAILLON-WITH-CHAINING board $(0, 1, 2, 1)$ (see **Figure 3.**). When we sow from pit $b_4$, our final seed is dropped in $b_3$, so our board is then $(0, 1, 3, 0)$. We have no other choice than to continue sowing in this chain, which lands in the Roumba, so $(1, 2, 0, 0)$ is a winning (non-chaining) TCHOUKAILLON board. In this case we only used one sequence of moves, but it is not uncommon to have a sequence of two or more moves which yields a winning board.

# 6 Minimum number of stones to obtain winning Tchoukaillon board

At this point, we might be wondering what the minimum number of stones required to produce a winning TCHOUKAILLON board is.

**Theorem 6.1.** *Fix $n \geq 0$. The $b_i(n)$ satisfy*

$$\sum_{j=1}^{i} b_j(n) \equiv n \pmod{i+1}$$

*, for each $i \geq 1$.*

Furthermore, we can uniquely determine the $b_i(n)$ using the latter theorem, which gives us

$$b_i(n) = \left( n = \sum_{j=1}^{i-1} b_j(n) \right) \pmod{i+1}.$$

We will use this in the proof of the next theorem.

**Theorem 6.2.** *Fix $k > 0$. A sequence of positive integers $(b_1, b_2, \ldots, b_k)$ represents a winning* TCHOUKAILLON *board iff for all $1 \le i \le k$ we have $b_i \le i$ and*

$$\sum_{j=i}^{k} b_j \equiv 0 \pmod{i}.$$

*Proof.* Suppose we have a sequence $(b_1, b_2, \ldots, b_k)$, and let $n = \sum_{j=1}^{k} b_j$. Let us now subtract the sum $\sum_{j=i}^{k} b_j$ from $\sum_{j=1}^{k} b_j \equiv n \pmod{i}$. We have that $(b_1, b_2, \ldots, b_k)$ satisfies $\sum_{j=1}^{i} b_j(n) \equiv n \pmod{i+1}$, for each $i \ge 1$. ☕

The following theorem presents a formula giving the minimum number of stones for a winning board.

**Theorem 6.3.** *The minimum number of stones $N(l)$ for a winning Tchoukaillon board of length $l$ is given by the formula*

$$N(l) = \frac{2}{1}\left\lceil \frac{3}{2}\left\lceil \cdots \left\lceil \frac{l-1}{l-2}\left\lceil \frac{l}{l-1}\right\rceil\right\rceil \cdots \right\rceil\right\rceil.$$

*Proof.* We begin by noting that for nonnegative integers $r$, $s$, and $k$, the next highest multiple of $k$ greater than or equal to $r$ is $k\lceil \frac{r}{k} \rceil$, and because $r < s$, we have $k\lceil \frac{r}{k} \rceil \le k\lceil \frac{s}{k} \rceil$. Using **Theorem 6.2**, we can construct a board of length $l$, which has the fewest number of seeds. Beginning with $b_l = l$, we can choose $b_i$ where $b_i + \sum_{j=i+1}^{l} b_j$ is the next highest multiple of $i$ greater than or equal to $\sum_{j=i+1}^{l} b_j$. Now, suppose that

$$\sum_{j=i+1}^{l} b_j = (i+1)\left\lceil \frac{i+2}{i+1}\left\lceil \cdots \left\lceil \frac{l-1}{l-2}\left\lceil \frac{l}{l-1}\right\rceil\right\rceil \cdots \right\rceil\right\rceil$$

and that $\sum_{j=i+1}^{l} b_j$ is as small as possible among boards of length $l$. Having chosen these $b_i$s,

$$\sum_{j=i}^{l} b_j = (i)\left\lceil \frac{i+1}{i}\left\lceil \cdots \left\lceil \frac{l-1}{l-2}\left\lceil \frac{l}{l-1}\right\rceil\right\rceil \cdots \right\rceil\right\rceil.$$

☕

# 7 The Chinese Remainder Theorem

As it turns out, some of the number theory involved with deriving winning TCHOUKAILLON boards creates a connection between TCHOUKAILLON and the Chinese Remainder Theorem. To review, the CRT reads as follows:

**Theorem 7.1.** *Let $n_1, \ldots, n_i$ be relatively prime integers, and let $a_1, \ldots, a_i$ be nonnegative integers such that $a_i < n_i$ for all $i$. Then, there is exactly one number $m < \prod_{j=1}^{i} n_j$ such that $m \equiv a_i \pmod{n_i}$ for each $i$.*

*Proof.* We need to prove both that such a number $m$ exists, and that it is unique. To show uniqueness, if there were two numbers $m_1$ and $m_2$ less than the product of the $n_i$ which satisfied this, because the $n_i$ are relatively prime, we would have that the product of the $n_i$ divides $m_1 - m_2$ (because $m_1$ and $m_2$ are congruent in each $\mathbb{Z}/n_i\mathbb{Z}$). But we must have that the difference $m_1 - m_2$ is less than this product, so it must be 0. Thus $m_1 = m_2$, and the proof is complete.

Now, let $P$ be the product of the $n_i$, and let $x_i$ be the remainder of $a_i \bmod n_i$. Consider the function $f(x) = (x_1, x_2, \ldots, x_i)$, where $x$ is a congruence class $\pmod{P}$. We claim that $f$ is bijective. Obviously, given a congruence class $\pmod{P}$, there is a unique set of congruence classes modulo each of the $n_i$, since $P$ is the product of the $n_i$. Moreover, from the last part of the problem we know that for every $i$-tuple of $x_i$, there is no more than 1 $x$ satisfying the equation, so the function is bijective as desired. However, it is easy to see that this function being bijective is equivalent to the CRT being true, so we are done. ☕

Now, to connect this theorem with TCHOUKAILLON, we will have to define the concept of a *remainder board*. This goes as follows:

**Definition 7.2.** A *remainder board* is an infinite board called $c(n)$ for some $n$. We set $c_i(n)$ to be the congruence class of $n$ in $\mathbb{Z}/i\mathbb{Z}$, and then $c(n) = (c_1(n), c_2(n)), c_3(n) \ldots$. A remainder board is said to be *increasing* if we choose the representative of each congruence class to be greater than or equal to the last number in the sequence. An increasing remainder board for $n$ is denoted $\widetilde{c}(n)$.

*Remark* 7.3. Note that at a certain point, once $i > n$, both the normal and increasing remainder boards will be constant because the congruence class of $n \pmod{i}$ will just be $n$.

*Example.* It is easy to calculate the value of $c(5)$, for instance: it is just $(1, 2, 1, 0, 5, 5, 5, \ldots$, and $\widetilde{c}(5) = (1, 2, 5, 5, 5, \ldots)$.

Now, we demonstrate the following result:

**Theorem 7.4.** *The value of $\widetilde{c}_i(n)$ is equal to $\sum_{j=1}^{i-1} b_i(n)$, where $b_i(n)$ is the number of stones in pit $i$ when the total arrangement has $n$ stones.*

*Remark* 7.5. Put into plain English, this theorem says that the number of stones in the $i^{\text{th}}$ pit of the remainder board of $n$ is equal to the sum of the first $i$ pits of the winning TCHOUKAILLON arrangement with $n$ stones.

*Proof.* For some $n$, we define $\widetilde{d}_i$ be $\widetilde{c}_{i+1}(n) - \widetilde{c}_i(n)$. Then, $\sum_{j=1}^{i-1} \widetilde{d}_i$ is clearly $\widetilde{c}_i(n) - \widetilde{c}_1(n)$, which is just $\widetilde{c}_i(n)$ because $\widetilde{c}_1(n) = 0$. In other words, the proof will be complete if we can just show that $\widetilde{d}_i = b_i(n)$. But this is true because $\widetilde{c}_i(n) \equiv i \pmod{n}$, and we saw that the $b_i(n)$ satisfy $\sum_{j=1}^{i+1} b_i(n) \equiv n \pmod{i}$ as well. In other words, the sum of $b_1$ through $b_i$ is congruent to the sum of $\widetilde{d}_1$ through $\widetilde{d}_i$ for all $i$ and a fixed $n \pmod{i}$. The result follows from a quick induction: if $\widetilde{b}_i = b_i$ is true for $i$ less than or equal to $k$, it must be true for $i = k + 1$ too or else it would disrupt the modular arithmetic equality. Thus, this proof is complete. ☕

With this, we are finally ready to prove a new result:

**Theorem 7.6.** *Let $m_{i_1}, \ldots, m_{i_k}$ be a sequence in $\mathbb{Z}^+$. We then have that the $m_{i_j}$ are all entries in some remainder board (i.e. each $m_{i_j} = \widetilde{c}_{i_j}(n)$ for some $n$ and all $m_{i_j}$) iff for all $i_p$ and $i_q$, $m_i \equiv m_j \pmod{\gcd(i, j)}$.*

*Remark* 7.7. This is essentially an analogue of the Chinese Remainder theorem for remainder boards, showing which kinds of sequences can be made into remainder boards. Because we have seen that remainder boards are so connected to winning TCHOUKAILLON games, though, it can also be used to recursively derive winning boards of this game.

*Proof.* For a given $n$, consider the remainder board $c_1(n), c_2(n), \ldots$. To prove the "only if" direction, assume that there are two value $i_p$ and $i_q$ such that $c_{i_p} \not\equiv c_{i_q} \pmod{\gcd(i_p, i_q)}$. This means that the value of $n$ mod $i_p$ and $n$ mod $i_q$ for a fixed $n$ are not congruent $\pmod{\gcd(i_p, i_q)}$. This can only hold true if $\gcd(i_p, i_q) > 1$. However, this contradicts basic modular arithmetic, so we have a contradiction. For the "if" direction of the biconditional, we just want to find a number $n$ such that $n \equiv m_{i_j} \pmod{i_j}$ for all $j$. But of course, the Chinese remainder theorem tells us that one will exist as long as the $i_j$ are relatively prime. The theorem statement did not assume that they were relatively prime, but this will only cause a problem if two of the congruence statements directly contradict each other, i.e. if the desired congruence classes of $n$ in $m_{i_p}$ and $m_{i_q}$ lead to incompatible results in mod $\gcd(i_p, i_q)$. But because we know that $m_{i_p}$ and $m_{i_q}$ are congruent $\pmod{\gcd(i_p, i_q)}$, this outcome will be impossible. Thus, the proof is complete. ☕

**Theorem 7.8.** *Consider a sequence $m_2, \ldots, m_k$ (note that we index the $m_i$ unconventionally to make the theorem statement more elegant), where each $m_i$ represent a congruence class $\pmod{i}$. Then, this sequence of numbers is equal to the number of stones in each pit of a winning TCHOUKAILLON board (i.e. for all $j$, $m_{j+1} = b_j(n)$) iff for all $i$ and $d \mid i$, $\sum_{j=i}^{i-d+1} \equiv 0 \pmod{d}$.*

*Proof.* We start with two definitions:

- An *allowable* sequence is one which which satisfies the given congruence for all $i$ and divisors $d$ of $i$.
- A *realizable* sequence is one which actually forms a list of entries of a winning TCHOUKAILLON board.

Now, our proof will operate by strong induction. The base case is $k = 3$, because then the sequence $m_i$ has only two members and so the sequence $m_2, m_3$ is vacuously allowable. Additionally, we can check by hand, that every possible value of $m_2$ and $m_3$ (keeping in mind that $m_2$ is 0 or 1 and $m_3$ is 0, 1, or 2) is realizable as well.

Moving onto the inductive step if the result holds for all $m_2, \ldots, m_k$, where $k$ is less than or equal to some number $x$, we will prove that it also holds for $k = x + 1$. To start with, consider a realizable sequence $m_2, \ldots, m_k$ (which must also be allowable), and consider what the value of $m_{k+1}$ could be while still maintaining a winning TCHOUKAILLON board. In other words, if we have that $b_1(n), b_2(n), \ldots, b_{k-1}(n)$ make up a winning TCHOUKAILLON board, and we want to know what $b_k(n)$ could be as $n$ ranges across all possible values. Note that by theorem $\boxed{4.3}$, the values of $b_k(n)$ are periodic with respect to $n$ with link $\mathrm{lcm}(2, 3, \ldots, k+1)$. However, the desired sequence of $m_i$ will only occur every $\mathrm{lcm}(2, 3, \ldots, k)$, which means as $n$ ranges over all positive integers we only have $Q = \frac{\mathrm{lcm}(2,3,\ldots,k+1)}{\mathrm{lcm}(2,3,\ldots,k)}$ different values of $m_{k+1}$ to worry about. Now, we claim that all values of $m_{k+1}$ which lead to a realizable sequence also lead to an allowable one. To start, because $m_2, \ldots, m_{k+1}$ is realizable, lemma $\boxed{4.1}$ tells us that

$$\sum_{i=2}^{k+1} m_i \equiv n \pmod{k+1}.$$

Then, letting $d$ be some factor of $k+1$, we can say that

$$\sum_{i=2}^{d} m_i + \sum_{i=d+1}^{k+1} m_i \equiv n \pmod{d}.$$

But again lemma $\boxed{4.1}$ tells us that $\sum_{i=2}^{d} m_j = \sum_{i=2}^{d-1} b_i(n) \equiv n \pmod{d}$, so we can simplify this equation to

$$\sum_{i=d+1}^{k+1} m_j \equiv 0 \pmod{d}.$$

Now, we have that $\sum_{i=2}^{d} m_i \equiv 0 \pmod{d}$, and $\sum_{i=d+1}^{k+1} m_i \equiv 0 \pmod{d}$, so adding these two equations together we have that $\sum_{i=2}^{k+1} m_i \equiv 0 \pmod{d}$ and the sequence is allowable by definition.

So now, we have shown that all realizable values of $m_{k+1}$ are allowable. To go the other direction, we will show that if $S_1$ is the set of all values of $m_{k+1}$ that are allowable, and $S_2$ is the set of all values that are realizable, $|S_1| = |S_2|$. We know that $|S_2| = Q = \frac{\mathrm{lcm}(2,3,\ldots,k+1)}{\mathrm{lcm}(2,3,\ldots k)}$, so we just need to prove that $|S_1| = Q$ as well. We do this by casework on the prime factorization of $k+1$:

- If $k+1$ is prime, we will have that all realizable values of $m_{k+1}$ are allowable because $k+1$ has no nontrivial factors, so the allowable condition is vacuously true (in other words, we can't even find $d$ to test it with).
- If $k = p^x$ for some prime $p$, then we can plug in $d = p^{x-1}$ into the allowable condition formula to fix the value of $m_{k+1} \pmod{p^{x-1}}$. This means that there are just $\frac{p^x}{p^{x-1}} = p$ different allowable values for $m_{k+1}$. Similarly, there are $Q = \frac{lcm(2,3,\ldots,p^x)}{\mathrm{lcm}(2,3,\ldots,p^{x-1})} = p$ *realizable* values here, so the $S_1 = S_2$ as desired.
- Finally, if $(k+1)$ is a composite number not equal to the power of a prime, let its prime factorization be $p_1^{i_1} p_2^{i_2} \ldots p_l^{i_l}$. Then, by setting $d = p_r^{i_r}$ for each $1 \leq r \leq l$, we can determine the congruence class of $m_{k+1}$ mod each factor of $k+1$ that is a prime power. Then, since all of the maximal prime powers in $k+1$'s prime factorization are coprime, the Chinese Remainder Theorem allows us to pick a unique $m_{k+1}$ satisfying these conditions and we will have that $Q = 1$ as well (because all of the factors of $k+1$ are already present in the numbers less than it). With this final step, we show that the two sets have equal size, and so the proof is complete.

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# 8 Conclusion

In this paper, we analyzed basic sowing-style games like MANCALA and ATOMIC WARI. Then, we turned our focus to the solitaire game of TCHOUKAILLON, and focused on how number theoretic properties determine the characteristics of its winning positions. We also analyzed several modifications of this game, such as TCHOUKAILLON-WITH-WRAPPING and TCHUKA ROUMBA. Finally, we used the Chinese Remainder Theorem to show which kinds of strings of numbers could be the values of pits on a winning TCHOUKAILLON board. In the future, hopefully more results will be achieved regarding how these results with solitaire sowing games apply to two-player games.

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