INTRODUCTION TO RICHMAN AND POORMAN GAMES

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Abstract. A bidding game is a combinatorial game where on each turn, the players bid for the right to move, and the specifics of the bidding rule vary from variant to variant. Arguably the two most common variants are the Richman and Poorman rulesets. In a Richman game, the winner of the bid pays their opponent. In a Poorman game, the winner of the bid pays a bank, which has no role in the game other than to collect funds from a winning bidder. We will be looking specifically at a game played with a token on a directed graph, where each player wants to move the token to a desired vertex.

1. Introduction

In this paper, we will introduce some bidding games and analyze some of their fundamental properties.

Definition 1.1. A bidding game is a combinatorial game defined as usual except instead of alternating moves, both players bid for the right to move.

Definition 1.2. In the Richman bidding game variant, the winner of the bid pays the loser.

These games are named after David Richman who studied them in the late 1980s.

To break ties, the players may alternate winning whenever there is a tie (the first player to win the tie decided arbitrarily), flip a coin, or add a ε chip whose owner is obligated to cast it in every bid. In this paper, however, we will not look into ties.

We consider the bidding game on a directed graph with a token, where Left wins if she can move the token, along edges of the graph, to a vertex b, and Right wins if he can get it to r. Although tactics like chance and bluffing may seem to be at play a large role in these games, there is actually usually a winning player who has a deterministic strategy.

Here is another variant of bidding games.

Definition 1.3. In the Poorman variant of a bidding game, the winner of the bid does not pay the amount paid to his/her opponent, rather the money is simply removed from the game.

This variant is interesting because on each turn the total chip supply decreases, leading to what seems like a more dynamic bidding aspect of the game. We discover that it can be analyzed similarly to a Richman game, however.

2. Richman Games on a Directed Graph

For all functions $f : V \to [0, 1]$, we define $f^+(v)$ as the maximum of $f(w)$ for successors w of v and $f^-(v)$ for the minimum.

Now we define a fundamental function for these games.

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Definition 2.1. The Richman cost function $R(v)$ satisfies $R(b) = 0, R(r) = 1$, and for all other vertices v, $R(v) = \text{avg}(R^+(v), R^-(v))$, where avg is the arithmetic mean.

Theorem 2.2. *Any directed graph has a Richman function.*

Proof. For nonnegative integers t, we can define $R(v, t)$ where $R(b, t) = 0$ and $R(r, t) = 1$ for all t, for $v \neq b, r$ we define $R(v, 0) = 1$ and recursively, $R(v, t) = \text{avg}(R^+(v, t-1), R^-(v, t-1))$.

Intuitively, this function approaches the Richman function we want for each iteration of t.

For each vertex v we take $R(v) = \lim_{t\to\infty} R(v, t)$. It is evident that if these limits exist, then $R(v)$ satisfies the conditions.

The limits exist because inductively, $R(v, t + 1) \leq R(v, t)$. Since the sequence $R(v, t)$ is non-strictly decreasing and bounded below by 0, it must be convergent. ∎

In fact, there is a *unique* Richman function for each directed graph, as we will show later.

The following surprising theorem shows why we care so much about Richman functions.

Theorem 2.3. *Consider the game where the token is on v, Left has L dollars, and Right has R dollars. If $\frac{L}{L+R} > R(v, t)$ then Left wins in at most t turns.*

Proof. For simplicity, we assume without loss of generality that $L + R = 1$.

We induct on t. For the base case of $t = 0$, the only possible case where $L > R(v, t)$ is if $v = b$, where Left has already won, so the statement is true.

Now we show the inductive step. Let u and w be vertices such that $R^-(v, t-1) = R(u, t-1)$ and $R^+(v, t-1) = R(w, t-1)$. I claim that Left can win by bidding $\text{avg}(R(u, t-1), R(w, t-1))$ dollars. If she wins, then she can move to vertex u with more than $R(u, t-1)$ dollars left and win in $t - 1$ turns by the inductive hypothesis.

If she loses the bid, then Right will move to some vertex x and Left will end up with more than $R(w, t - 1)$ dollars. Since this is greater than $R(x, t - 1)$, Left wins in $t - 1$ turns by the inductive hypothesis. ∎

Corollary 2.4. *If $\frac{L}{L+R} > R(V)$, then Left has a winning strategy.*

This follows easily from Theorem 2.3 by taking t to infinity.

We would like to derive a version of Theorem 2.3 for Right, but the symmetry does not immediately follow. Since we arbitrarily set $R(b) = 0$ and $R(r) = 1$, we might believe that if $\frac{R}{L+R} > 1 - R(v)$, or equivalently $\frac{L}{L+R} < R(v)$ then Right has a winning strategy. However, the proof of Theorem 2.3 actually cannot be symmetrically argued for Right because the function $R(v, 0)$ was not defined symmetrically, causing the base case to not work out. We defined $R(v, 0) = 1$ for all $v \neq b, r$, not 0. This motivates us to define $r(v, t)$ the same fashion as $R(v, t)$ except $r(v, t) = 0$ for all $v \neq b, r$. Now, with the same proof as Theorem 2.3, we conclude that Right wins if $\frac{R}{L+R} > 1 - r(v)$. Note that r is another Richman function. Amazingly, we can show that it must be equal to R.

Theorem 2.5. *The Richman function of a directed graph is unique.*

Proof. The following argument is rather technical.

Definition 2.6. If u is the vertex such that $R^-(v) = R(u)$, then we call the edge between v and u the *edge of steepest descent.*

Definition 2.7. We let $\overline{v}$ be the maximal set of vertices $v = v_0, v_1, \ldots, v_k$ such that the vertex between v_i and v_{i+1} is the edge of steepest descent.

Here is the key lemma.

Lemma 2.8. *For any Richman function R, if $R(v) < 1$ then $b \in \overline{v}$.*

Proof. Take the vertex $a \in \overline{v}$ such that $a = \min_{u \in \overline{v}} R(u)$. Note that if $a = b$, then we are done. Otherwise, note that $R^-(a) = R(u) = R(a)$ by how we defined $\overline{v}$. Thus, u satisfies the same property as a did. We can keep choosing successors in this way. Note that r cannot be reached this way because $R(r) = 1 > R(a)$, so we must instead reach $b \in \overline{v}$. ∎

Now we are ready to prove Theorem 2.5. Suppose we have two Richman functions R_1 and R_2. Let v be the vertex that maximizes $R_1(v) - R_2(v)$.

Now define u_1, u_2, w_1, w_2 such that $R_i^-(v) = u_i$ and $R_i^+(v) = w_i$.

By definition, $R_1(u_1) \leq R_1(u_2)$, so

$$R_1(u_1) - R_2(u_2) \leq R_1(u_2) - R_2(u_2) \leq M,$$

where the second inequality follows from the definition of M. Similarly, we have

$$R_1(w_1) - R_2(w_2) \leq R_1(w_1) - R_2(w_1) < M.$$

Adding the two centered inequalities gives

$$R_1(u_1) + R_1(w_1) - R_2(u_2) - R_2(w_2) \leq 2M.$$

But notice that the left-hand side is $2M$ by our definition of Richman functions, so equality must hold in each of the centered inequalities.

In particular, we have $R_1(u_2) - R_2(u_2) = M$, so u_2 satisfies the same condition as v. By induction, $R_1(u) - R_2(u) = M$ for all $u \in \overline{v}$, where $\overline{v}$ is with respect to R_2. WLOG, $v \neq r$, since there is always another choice for v besides r. Then by our lemma, $b \in \overline{v}$, so $M = R_1(b) - R_2(b) = 0$. Thus, $M = 0$ and we have $R_1(v) - R_2(v) \leq 0$ for all vertices v. Symmetrically, we can show $R_2(v) - R_1(v) \leq 0$, so $R_1(v) = R_2(v)$ everywhere, as desired. ∎

Hence, $R(v) = r(v)$ everywhere. Combined with Corollary 2.4, this means Left wins if her share of the total money exceeds $R(v)$ and loses if her share of the total money is below $R(v)$. If her share of money is equal to $R(v)$, we unfortunately cannot conclude with certainty who wins.

3. Poorman Games on a Directed Graph

Even though the structure of the game seems to be significantly different because the total money supply dwindles as the game progresses, the analysis of Poorman games is actually rather similar.

When working with Poorman games, we will use another notion of "average", defined as follows.

Definition 3.1. For any reals $0 \leq x \leq y \leq 1$, we say the Poorman average of x and y is $\text{avg}_P(x, y) = \frac{y}{1-x+y}$.

It is easy to check that $x < \text{avg}_P(x, y) < y$, so Poorman averages behave as we expect them to.

As expected, we define the Poorman cost function as we defined the Richman cost function, except with the Poorman average instead of a regular average (arithmetic mean).

Definition 3.2. The Poorman cost function $P(v)$ over a graph satisfies $P(b) = 0, P(r) = 1$, and for all other vertices v, $P(v) = \text{avg}_P(P^+(v), P^-(v))$.

The Poorman cost function satisfies many of the same properties Richman functions in their respective contexts. Many of the following results will be familiar from the previous section.

Theorem 3.3. *There exists a Poorman cost function for any directed graph.*

Proof. Very similar to the proof for the Richman function. Define $P(v, t)$ as $R(v, t)$ except with the Poorman average instead of regular average.

Inductively, $P(v, t + 1) \le P(v, t)$, so $P(v) = \lim_{t\to\infty} P(v, t)$ exists and satisfies the properties of a Poorman cost function. ■

Theorem 3.4. *In a Poorman game with token on vertex v, Left wins if $\frac{L}{L+R} > P(v)$.*

Proof. This is similar to the corresponding proof for Richman functions (induction on t), except the underlying algebra is more complex. The algebra works out due to the way we defined the Poorman average.

Unfortunately, since the total money supply is no longer constant, assuming $L + R = 1$ isn't very useful here, so we will opt not to assume that.

For the base case the only way for $\frac{L}{L+R} > R(v, 0)$ is if $v = b$, where Left already won.

Now we show the inductive step. I claim that Left wins by bidding

$$B(v) = L \cdot \frac{P(v,t) - P^-(v,t-1)}{P(v,t)(1 - P^-(v,t-1))} = L \cdot \frac{P^+(v,t-1) - P(v,t)}{P(v,t)P^+(v,t-1)}.$$

If Left wins, then the proportion of her money to the total money is at least

$$\frac{L - B}{L + R - B} > \frac{L - B}{\frac{L}{P(v,t)} - B} = P^-(v, t-1).$$

Thus, Left wins in $t - 1$ more moves by the inductive hypothesis.

If Left loses, then her share of the money is at least

$$\frac{L}{L + R - B} > \frac{L}{\frac{L}{P(v,t)} - B} = P^+(v, t-1)$$

, so Left wins in $t - 1$ turns by the inductive hypothesis. ■

Now we will show that Poorman functions are unique. The proof is a highly interesting one in terms of winning strategies!

Theorem 3.5. *The Poorman function of a directed graph is unique.*

Proof. We will accomplish this by proving that *any* Poorman cost function has the property that Left wins if $\frac{L}{L+R} > P(v)$. Then symmetrically, Right wins if $\frac{R}{L+R} > 1 - P(v)$ or equivalently $\frac{L}{L+R} < P(v)$. Clearly, there can be only one function that satisfies these properties, as both players cannot both have a winning strategy for the same game.

Note that we did not prove this already in Theorem 3.4 because that proof was only for the Poorman function defined by $P(v) = \lim_{t\to\infty} P(v, t)$.

Definition 3.6. Left's *surplus* is

$$\varepsilon = L - \frac{P(v)R}{1 - P(v)}.$$

Note that if $\frac{L}{L+R} > P(v)$ then ε is positive.

The general idea is that Left can put this surplus into her "slush fund", with $L' = L - \varepsilon$ left in her main balance. We show that $L' = L - \varepsilon$ money is enough to avert a loss indefinitely, and with the extra ε Left can win. Like in the proof of Theorem 3.4, we define

$$B(v) = L' \cdot \frac{P(v,t) - P^-(v,t-1)}{P(v,t)(1 - P^-(v,t-1))} = L' \cdot \frac{P^+(v,t-1) - P(v,t)}{P(v,t)P^+(v,t-1)}.$$

Left's strategy is to bid $B(v) + \alpha$ for some strategic value of α.

If Left wins the bid, then she pays $B(v)$ from her main balance and α from her slush fund. It can be checked that

$$\frac{L' - B(v)}{L' + R - B(v)} = P^-(v),$$

so Left's portion of the total money excluding the slush fund remains at the critical value.

If Right wins the bid, then suppose she moves to w. Since

$$\frac{L'}{L' + R - B(v)} = P^+(v) \geq P(w)$$

and Right also paid at least α more than $B(v)$, so Left's can put $\frac{\alpha P(w)}{1-P(w)}$ more money into her slush fund and still have more money than the critical amount. Thus it is impossible for $P(w) = 1$, so Right cannot win on this move. This already shows that Left can avert a loss indefinitely.

Left's strategy to win is to pick an increasing series of choices of investments $\alpha_1, \ldots, \alpha_n$ such that she has enough money in her slush fund to pay for these investments and if Right wins a move stopping this series of investments, Left gets to put more money into her slush fund than she spent on investments.

We let m be the smallest non-zero value of $\frac{P(v)}{1-P(v)}$ and $r = 1 + \frac{2}{m}$. Then we set $a_1 = \frac{2\varepsilon}{m(r^n-1)}$ and $a_i = a_1 r^{i-1}$. By the geometric series formula, we can check that

$$a_1 + a_2 + \cdots + a_n = a_1 \cdot \frac{r^n - 1}{r - 1} = \varepsilon.$$

Now suppose Right wins on the ith bid in this sequence. Then Left's slush fund increases by at least ma_i since $ma_i < a_i \cdot \frac{P(v)}{1-P(v)}$ by definition. Left had invested from her slush fund

$$\begin{aligned} a_1 + \cdots + a_{i-1} &= a_1 \cdot \frac{r^{i-1} - 1}{\frac{2}{m}} \\ &\leq \frac{m}{2} a_1 r^{i-1} \\ &= ma_i/2, \end{aligned}$$

so Left makes a net profit of at least $ma_i - ma_i/2 \geq ma_1/2$.

Thus her slush fund is increased by at least a factor of

$$\frac{\varepsilon + ma_1/2}{\varepsilon} = 1 + \frac{1}{\left(1 + \frac{2}{m}\right)^n - 1}$$

every n moves. But Left's slush fund is bounded by L, her original amount of money. Thus, it cannot increase forever, so Left must eventually win. ∎

Therefore, just like with Richman games, for every Poorman game there is a critical value $P(v)$ for which Left wins if her proportion of money exceeds $P(v)$, and Left loses if it is lower than $P(v)$.

References

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[LLPU96] Andrew J. Lazarus, Daniel E. Loeb, James G. Propp, and Daniel H. Ullman. Richman games. *Games of No Chance*, 29, 1996. URL: `http://library.msri.org/books/Book29/files/propp.pdf`.