AN OVERVIEW OF SURREAL NUMBERS

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Abstract. The surreal numbers are constructed in stages, with each surreal number being defined by two other surreal numbers constructed earlier. For two sets of numbers G^L and G^R, the number $\{x \mid y\}$ is a surreal number if every $x \in G^L$, every $y \in G^R$, and $x < y$. In this paper, we explore the structure of surreal numbers and related objects such as generalized (or omnific) integers and ordinals. Additionally, we give an exposition on the structure of the class of surreal numbers, the applications of surreal numbers to combinatorial game theory, as well as on some objects inspired by surreal numbers such as pseudo numbers. We also provide a brief introduction to the Field $\mathbf{On}_2$ and its properties.

1. A Brief History and Introduction

Surreal numbers are fascinating for several reasons. They are built on an extremely simple and small foundation, and yet they provide virtually all of the capabilities of ordinary real numbers. With surreal numbers we are able to (or rather, required to) actually prove things we normally take for granted, such as $x = x$ or $x = y$ implies $x + z = y + z$. Furthermore, surreal numbers extend the real numbers with a tangible concept of infinity and infinitesimals (numbers that are smaller than any positive real number, and yet are greater than zero).

Surreal numbers were popularized by Donald Knuth's (fiction) book *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness*, and the full theory was developed by John Conway after using the numbers to analyze endgames in Go. In [2] Conway himself said that "I walked around for about six weeks after discovering the surreal numbers in a sort of permanent daydream, in danger of being run over." This sense of reverie overtakes others who study them. Martin Kruskal, a mathematician of wide-ranging achievements, spent some of his later years studying the surreal numbers, and he wrote "The usual numbers are very familiar, but at root they have a very complicated structure. Surreals are in every logical, mathematical, and aesthetic sense better."

The surreal numbers form a field, which is to say that they can be added, subtracted, multiplied and divided, so long as you do not try to divide by 0. They include the familiar real numbers as a tiny subfield, and like the reals they are a linearly ordered field. So far so unremarkable, but they also include the transfinite ordinals; and since they are a field, they include, along with the first infinite ordinal ω, such wonders as $\omega - 1$, not to mention $\omega/2$ and $2/\omega$, and they also include $\sqrt{\omega}$ and ω^n for any real number, and for any surreal number.

What can surreal numbers be used for? Not very much at present, except for some use in combinatorial game theory. But it is still a new field, and the future may show uses that we haven't thought of. Nevertheless, surreal numbers are worth studying for two reasons. First, as a study in pure math they are a fascinating—even beautiful—subject. Before we start looking at the definition, you must forget every thing you know about numbers (more precisely inequalities concerning with real numbers or integers). Every surreal number is created on a certain day and corresponds to two sets of numbers. For a surreal number, x, we write $x = \{G^L \mid G^R\}$ and call G^L and G^R the left and right set of x.

Definition 1.1. Let $x = \{G^L \mid G^R\}$and $y = \{H^L \mid H^R\}$. Then we say that $x \geqslant y$ if for every $x^R \in G^R, x^R > y$, and for every $y^L \in H^L, x > y^L$. We say that $x > y$ if $x \geqslant y$ and $y \ngeqslant x$, and $x = y$ if $x \geqslant y$ and $y \geqslant x$.

Now we can define Surreal numbers in a more precise way as follows:

Definition 1.2. Suppose that G^L and G^R are two sets of surreal numbers (or simply numbers). If for every $x \in G^L$ and $y \in G^R$ we have $x < y$, then we say that $\{G^L \mid G^R\}$ is a surreal number.

2. The Structure of **NO**

The class **NO** of surreal numbers and the surreal numbers themselves contain many interesting properties, some of which we explore in this section. We first describe the method of constructing the class **NO**, after which we examine some of the characteristics of general surreal numbers.

Definition 2.1. A Dedekind cut or Dedekind section is a set partition of the rational numbers into two nonempty subsets S_1 and S_2 such that all members of S_1 are less than those of S_2 and such that S_1 has no greatest member.

Definition 2.2. If L and R be subsets of x such that $L < R$; then (L, R) will be called a Conway cut in x. Note that if (L, R) is a Conway cut in x, then L or R may be empty. Further, note that the union of L and R may be a proper subset of x.

It is known that the real numbers $\mathbb{R}$ can be constructed in this way. Conway's method of building up number systems may be regarded as Dedekind cuts taken to extremes. His basic principle states that if L, R are two sets of numbers and no member of L is greater than any member of R, then $\{L \mid R\}$ is a number. All numbers are constructed in this way.

One of Conway's brilliant ideas is that of the birth order of surreal numbers. Starting from $0 = \{|\}$ we get all dyadic numbers and, transfinitely, one can get all real numbers. However we can go even further and construct a Field **NO**, whose members are surreal numbers.

Definition 2.3. Let $\alpha, \beta, \gamma, \cdots$ denote arbitrary ordinals. For each $\mathcal{A} \in \{\alpha, \beta, \gamma, \cdots\}$ let

1. $O_{\mathcal{A}}$ be the set of all numbers born before day $\mathcal{A}$.
2. $M_{\mathcal{A}}$ be the set of all numbers born on or before day $\mathcal{A}$.
3. $N_{\mathcal{A}}$ be the set of all numbers born on $\mathcal{A}$.

Each $x \in N_{\mathcal{A}}$ defines a Dedekind section L, R of $O_{\mathcal{A}}$, setting

$$L = \{y \in O_{\mathcal{A}} | y < x\} \quad \text{and} \quad R = \{y \in O_{\mathcal{A}} | y > x\}$$

gives, by the simplicity theorem, $x = \{L \mid R\}$; thus $M_{\mathcal{A}} = O_{\mathcal{A}} \cup N_{\mathcal{A}}$. Let $x \in N_{\mathcal{A}}$, then for each $\mathcal{A} < \mathcal{A}$, x defines a section in $O_{\mathcal{A}}$, which defines a unique point $x_{\mathcal{A}} \in N_{\mathcal{A}}$.

Definition 2.4. Define $x_{\mathcal{B}}$ to be the $\mathcal{B}^{\text{th}}$ approximation to x. Moreover $x_{\mathcal{B}} = x \; \forall \; \mathcal{B} \geq \mathcal{A}$.

Theorem 2.5. *Every number x is in a unique set $N_{\mathcal{A}}$.*

Proof. Assume that this holds for all x^L, X^R. If $\mathcal{B}$ is an ordinal greater than the birthdays of all x^L, x^R, then $x \in M_{\mathcal{B}}$, and thus $x \in N_{\mathcal{A}}$ for some $\mathcal{A} \leq \mathcal{B}$. ∎

Definition 2.6 (Sign-Expansion)**.** For each $\mathcal{B} < \mathcal{A}$ (the birthday of x) let $s_{\mathcal{B}}$ ($+$ or $-$) be the sign of the number $x - x_{\mathcal{B}}$. And $s_{\mathcal{B}} = 0 \;\forall\; \mathcal{B} \geq \mathcal{A}$.

Thus we assign to each x, below some ordinal, 0 beyond, a sequence of signs $+$ or $-$ where we consider, $- < 0 < +$. And $(s) < (t)$ iff for some $\mathcal{A}$ we have

$$s_{\mathcal{B}} = t_{\mathcal{B}} \;\forall\; \mathcal{B} < \mathcal{A} \text{ and } s_{\mathcal{A}} < t_{\mathcal{A}}.$$

Theorem 2.7. *Let x and y have sign-expansions (s) and (t). Then we have $x < y$, $x = y$, $x > y$ according as $(s) < (t)$, $(s) = (t)$, $(s) > (t)$.*

Proof. If $(s) < (t)$, suppose $s_{\mathcal{B}} = t_{\mathcal{B}} \;\forall\; \mathcal{B} < \mathcal{A}$, but $s_{\mathcal{A}} < t_{\mathcal{A}}$. Then $x_{\mathcal{A}} = y_{\mathcal{A}}$ by induction for all $\mathcal{B} < \mathcal{A}$, but $x_{\mathcal{A}} < y_{\mathcal{A}}$. The sections defined by x and y in $O_{\mathcal{A}}$ now show that $x < y$. If $(s) = (t)$, we find that x and y define the same section of $O_{\mathcal{A}}$, where $\mathcal{A}$ is their common birthday, and so $x = y$. ∎

Theorem 2.8. *For an arbitrary sequence (s) of signs $+$ or $-$ below some ordinal $\mathcal{A}$,* 0 *beyond, there exists a number x whose sign-expansion is (s).*

As a consequence we see that there exists a bijection between numbers and their sign-expansions which is monotonous.

A quick application of sign-expansions is a method for computing the value of a real number, found by Berlekamp. Assume the grounded edge is Left's. If all edges in the string are Left's, the value is clearly an integer equal to the number of edges. Otherwise identify the first left-right alternation. Left's edges before the alternation contribute 1 each. Replace the two alternating edges by a decimal point and replace each subsequent left (respectively, right) edge by a 1 (respectively, 0) and append a 1. You can now read off the fractional value in binary. This can be used to get real multiples of ω as $+^{\omega} -^{\omega} -^{\omega}$ is the sign-expansion for $\frac{1}{4}\omega$. However can also be used to determine the value of a Hackenbush stalk.

For an example consider the Hackenbush stalk in Figure 1. This is a string of

$$\underbrace{\text{R R R R}}_{-4}\quad \underbrace{\text{R L}}_{\bullet}\quad \underbrace{\text{L}}_{0}\quad \underbrace{\text{R}}_{1}\quad \underbrace{\text{R}}_{1}\quad \underbrace{\text{L}}_{0}\quad \underbrace{\text{R}}_{1}\quad \underbrace{\text{L}}_{0}$$

which gives

$$-\left(4 + \frac{1}{4} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128}\right) = -4\frac{53}{128}.$$

However, there exists another method to find the value of a Hackenbush Stalk, and more generally a real number, proposed by Thea van Roode in [6]. First assign value 1 to edges until the first color change. Thereafter, divide by 2 at each new edge. The sign of each edge depends on its color. The reader is encouraged to prove that both of these methods work and to calculate the value of the Hackenbush stalk in Figure 1 using Thea van Roode's method.

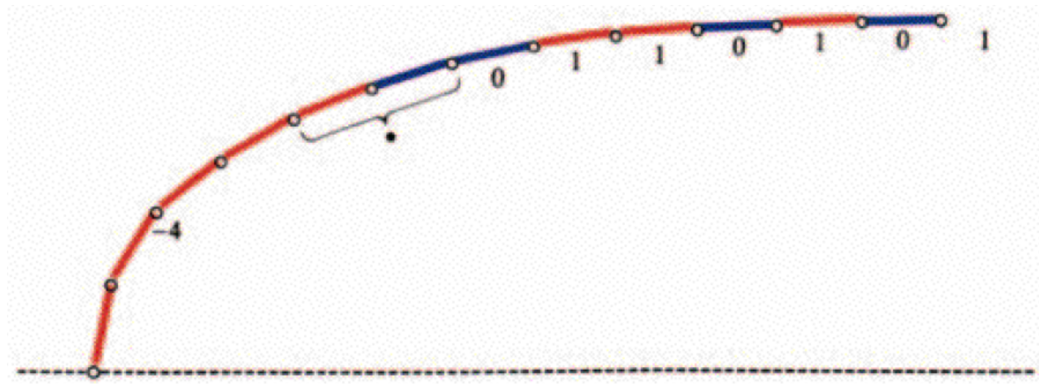

Figure 1. Using Berlekamp's rule for a HACKENBUSH Stalk

2.1. The ω-map.

Definition 2.9. Call positive numbers x and y commensurate if for some positive integer n we have $x < ny, y < nx$.

Definition 2.10. Let x be a surreal number. Then we define

$$\omega^x = \left\{0, r\omega^{x^L} \;\middle|\; r\omega^{x^R}\right\}$$

where r ranges over all positive real numbers.

Theorem 2.11. *Each positive number is commensurate with some of ω^y.*

Proof. Let $x = \{0, x^L \mid x^R\}$ with $x^L, x^R > 0$. Each x^L is commensurate with some ω^{y^L} (say) and each x^R with ω^{y^R}. If x is commensurate with one of its options, we are done. If not, we can add all numbers $r\omega^{y^L}$ as Left options and all $r\omega^{y^R}$ as Right options, and we then see that $x = \omega^y$, where y is the number $\{y^L \mid y^R\}$. ■

Theorem 2.12. *We have that $\omega^0 = 1$ and $\omega^{-x} = \frac{1}{\omega^x}$. But importantly $\omega^x \cdot \omega^y = \omega^{x+y}$.*

Proof. We omit the proof here; take a look at [2] for a formal proof. ■

Thus we see that ω^x has properties of an xth power of ω.

2.2. The Normal Form of x.

Now that we know how to raise ω to surreal number powers, we can define the Conway normal form, a generalization of the Cantor normal form for ordinal numbers.

Theorem 2.13 (Conway normal form)**.** *We can express each surreal number x uniquely in the form*

$$x = \sum_{\mathcal{B}<\mathcal{A}} \omega^{y_{\mathcal{B}}} \times r_{\mathcal{B}}$$

where the numbers $r_{\mathcal{B}}$ are nonzero real numbers, and the $y_{\mathcal{B}}$'s are a decreasing sequence of surreal numbers (i.e. if $\mathcal{B} < \mathcal{B}' < \mathcal{A}$, then $y_{\mathcal{B}} > y_{\mathcal{B}'}$).
Normal forms for distinct x are distinct, and every form satisfying these conditions occurs.

Proof. For a proof look at [2]. ■

2.3. Sign-Expansions and Normal Form.

Definition 2.14. Call irrelevant the sign Y_δ in the sign-expansion of y if the number with sign-expansion

$$[Y_0, \ldots, Y_z, \ldots]_{z<\delta}$$

is greater than or equal to some $x > y$ with $r_x \neq 0$.

Definition 2.15. The relevant sign-expansion of y is that obtained by omitting all the irrelevant signs from its ordinary sign-expansion.

Let our number be of the form $\omega^x \cdot r + \omega^y \cdot s + \omega^z \cdots t + \ldots$ Suppose that $x, y, z, \ldots$ have relevant sign-expansions

$$[X_\delta]_{\delta<\mathcal{A}}, [Y_\delta]_{\delta<\mathcal{B}}, [Z_\delta]_{\delta<\mathcal{C}}, \ldots$$

and that $r, s, t, \ldots$ have ordinary sign-expansions

$$[R_0, R_1, \ldots], [S_0, S_1, \ldots], [T_0, T_1, \ldots], \ldots.$$

Then the sign-expansion of our number is

$$[(X_0 R_0)^{\omega^{e_0+1}}, \ldots, (X_\delta R_0)^{\omega^{e_\delta+1}}, \ldots, R_1^{\omega^{e_\mathcal{A}}}, R_2^{\omega^{e_\mathcal{A}}}, \ldots,$$
$$(Y_0 S_0)^{\omega^{f_0+1}}, \ldots, (Y_\delta S_0)^{\omega^{f_\delta+1}}, \ldots, S_1^{\omega^{f_\mathcal{B}}}, S_2^{\omega^{f_\mathcal{B}}}, \ldots,$$
$$(Z_0 T_0)^{\omega^{g_0+1}}, \ldots, (Z_\delta T_0)^{\omega^{g_\delta+1}}, \ldots, T_1^{\omega^{g_\mathcal{C}}}, T_2^{\omega^{g_\mathcal{C}}}, \ldots],$$

where for each $\delta < \mathcal{A}$, e_δ denotes the ordinal number of $+$ signs among the numbers $X_\varepsilon(\varepsilon < \delta)$, and the numbers $f_\delta, g_\delta, \ldots$ are defined similarly for the numbers $y, z, \ldots$.

2.4. On Numbers given by Refinements of (Timely) Conway Cuts.

We return to our study of Conway cuts and make refinements.[1]

Definition 2.16. A Conway cut representation (L, R) of x will be called timely if L and R are subsets of $0_\mathcal{A}$. The only timely representation of 0 is $\{|\}$.

Theorem 2.17. *Each $x \in NO$ has a timely representation.*

Proof. For a proof look at [1]. ∎

Theorem 2.18. *Let (L, R) and (L', R') be timely Conway cut representations in NO, such that $\{L \mid R\} = \{L' \mid R'\}$ then (L, R) and (L', R') are equivalent.*

Proof. For a proof look at [1]. ∎

Definition 2.19 (Refinements)**.** [1] Let L' and R' be subsets of **NO** such that

1. $L' < \{x\} < R'$
2. for all $x^L \in L$ there exists $x^{L'} \in L'$ such that $x^L \leq x^{L'}$.
3. for all $x^R \in R$ there exists $x^{R'} \in R'$ such that $x^{R'} \leq x^R$.

Then (L', R') will be called a refinement of (L, R).

Theorem 2.20. *Let (L', R') be a refinement of (L, R), then $\{L' \mid R'\} = \{L \mid R\}$*

[1] Get the pun now?

Proof. Let

$$I = \{y \in \mathbf{NO} \mid L < \{y\} < R\}, \text{ and } I' = \{y \in \mathbf{NO} \mid L' < \{y\} < R'\}.$$

By definition we see that $x \in I'$ and that I' is a sub-interval of the interval I. Since x is the simplest element in I, it is certainly the simplest element of I'. ∎

Theorem 2.21. *Let $y = \left\{y^L \mid y^R\right\} \in \mathbf{NO}$, and let $(y^{L'}, y^{R'})$ be a refinement of (y^L, y^R). Then*

$$x + y = \left\{x^{L'} + y, x + y^{L'} \,\middle|\, x^{R'} + y, x + y^{R'}\right\}$$

and

$$xy = \left\{x^{L'}y + xy^{L'} - x^{L'}y^{L'}, x^{R'}y + xy^{R'} - x^{R'}y^{R'} \,\middle|\, x^{L'}y + xy^{R'} - x^{L'}y^{R'}, x^{R'}y + xy^{L'} - x^{R'}y^{L'}\right\}.$$

Proof. For a proof look at [1]. ∎

2.5. **Irreducible numbers.** One can ask an interesting question on whether any index in the normal form of $\mathcal{A}$ have the same birthday as $\mathcal{A}$, if not then the normal form yields an expression for x in terms of (real and ordinal numbers and) simpler numbers. We call such number x as *irreducible*. Now suppose the index $y_{\mathcal{A}}$ in the $\mathcal{A}$–term of x has the same birthday as x. Then it is easy to see that $\omega^{y_{\mathcal{A}}} \cdot r_{\mathcal{A}}$ is the last term in the normal form of x and that $r_{\mathcal{A}} = \pm 1$. Note that this is followed simply because the numbers

$$\sum_{\mathcal{B}<\mathcal{A}} \omega^{y_{\mathcal{B}}} \cdot r_{\mathcal{B}} \pm \omega^{y_{\mathcal{A}}}$$

are constructed strictly before

$$\sum_{\mathcal{B}<\mathcal{A}} \omega^{y_{\mathcal{B}}} \cdot r_{\mathcal{B}} + (\omega^{y_{\mathcal{A}}} \cdot r_{\mathcal{A}} + \text{smaller})$$

So in the case of irreducible numbers we can write $x = X \pm \omega^{y_{\mathcal{A}}}$, where X is born before x. Note that ω^y is quite small compared to x, and y has the same birthday as x. Thus if y is reducible in the sense above, then by inserting the normal form for y, we obtain an expression for x in terms of simpler numbers, and so as earlier we regard x as reducible. The irreducible numbers generalize the concept of $\mathcal{E}$–numbers, and it is not hard to see that the birthday of any irreducible number is an $\mathcal{E}$–number.

2.6. **Continued Exponential for Irreducibles.** The continued exponential expression for the number x mentioned in 2.5 is

$$x = x' \pm \omega^{y' \pm \omega^{z' \cdot^{\cdot^{\cdot}}}}$$

which we write as

$$x = x' \pm \omega^{y'^{\pm}} \omega^{z'^{\pm}} \omega^{\cdots}.$$

Note that there exists multiple numbers with the same continued exponential thus we cannot uniquely determine x. Let E be

$$a \pm \omega^{b^{\pm}} \omega^{c^{\pm}} \omega^{\cdots}.$$

Let E_0 be the first number to be born with this as its continued exponential. And then we get E_{-1} and E_1 to the left and right of E_0, respectively. And $E_{\frac{1}{2}}$ lies between E_0 and E_1. And similarly define E_x for every number x.

Let

$$\varepsilon = \omega^{\omega^{\cdot^{\cdot^{\cdot}}}}.$$

Then ε_0 denotes the first ordinal ε-number greater than ω and ε_1 denotes the next ε-number. Respectively we have

$$\varepsilon_0 = \left\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots \,\middle|\,\right\} \text{ and } \varepsilon_1 = \left\{\varepsilon + 1, \omega^{\varepsilon+1}, \dots \,\middle|\,\right\},$$

and so on.

Now look at the number

$$\delta = \{\text{ordinals} < \varepsilon \mid \varepsilon - 1, \omega^{\varepsilon-1}, \omega^{\omega^{\varepsilon-1}}, \dots\}.$$

It is easy to see that $\delta = \omega^\delta$ and thus we get the continued exponential expression of δ to be $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$. And as δ is the first number constructed left of ε with this expression, we get $\delta = \varepsilon_{-1}$. Similarly

$$\varepsilon_{-\frac{1}{2}} = \left\{\delta + 1, \omega^{\delta+1}, \omega^{\omega^{\delta+1}}, \dots \,\middle|\, \varepsilon - 1, \omega^{\varepsilon-1}, \omega^{\omega^{\varepsilon-1}}, \dots\right\}.$$

Theorem 2.22. *The solutions to the equation $x = \omega^x$ are the ε-numbers. Also the equation $x = \omega^{-x}$ has a unique solution. Thus $x = \omega^{-y}$ and $y = \omega^{-x}$.*

2.7. **Functions on NO.** For a subclass $A \subset \mathbf{NO}$, functions of the form $f : A \to \mathbf{NO}$ map each $x \in A$ to some $f(x) \in \mathbf{NO}$. One important prerequisite of studying surreal analysis (see [5]) is to come up with the notion of continuity of functions over the surreal numbers; to do this, we define a topology on **NO**.

Definition 2.23. A *topology* on **NO**, as defined in [4], is a collection of sub-classes $\mathcal{C}$ of **NO** that satisfy the following three properties.

(1) $\varnothing$ and **NO** are in $\mathcal{C}$.
(2) $\bigcup_{i\in I} A_i \in \mathcal{C}$ for any sub-collection $\{A_i\}_{i\in I} \subset \mathcal{C}$ indexed over a proper set I.
(3) $\bigcap_{i\in I} A_i \in \mathcal{C}$ for any sub-collection $\{A_i\}_{i\in I} \subset \mathcal{C}$ indexed over a finite set I.

Before we formally define continuity of surreal functions, we need to get an idea of what it means for a subclass of **NO** to be open.

Definition 2.24. We say that a nonempty subinterval of **NO** is open if it has endpoints in $\mathbf{NO} \cup \{\mathbf{On}, \mathbf{Off}\}$ (where **On** is the class of ordinals and $\mathbf{Off} = -\mathbf{On}$) and does not contain the endpoints of **NO**. The subclass $A \subset \mathbf{NO}$ is open if it can be written as $A = \bigcup_{i\in I} A_i$ where I is a proper set and each A_i is an open subinterval of **NO**.

Thus with the definition of open subclasses, we have completed our definition of a topology on **NO**. For a complete proof of this, see [4]. Finally, we can define what it means for surreal functions to be continuous.

Definition 2.25. For $A \subset \mathbf{NO}$, let $f : A \to \mathbf{NO}$ be a function. We say that f is continuous on A if for any open class $B \in \mathbf{NO}$, $f^{-1}(B)$ is open in A.

With this idea of continuity of surreal functions in hand, we have the necessary tools we need to study surreal analysis, the surreal analogue to real analysis. We will not be doing any surreal analysis here, but we encourage the reader to take a look at [5].

3. The Class **Oz** of Omnific Integers

Now that we have explored some of the underlying structure of surreal numbers, we can look at the surreal number analogue to integers: omnific integers. The literature regarding these numbers is limited; the main results in this section can be attributed to Conway who wrote about these numbers in [2].

Definition 3.1. The *omnific* (or *generalized*) *integers*, first defined by Simon P. Norton, are numbers of the form

$$x = \{x - 1 \mid x + 1\}.$$

The class consisting of these omnific integers is known as **Oz**.

Definition 3.2 (Conception Day)**.** The *conception day* of a surreal number x, denoted $\kappa(x)$, is the smallest ordinal that is the birthday of a number equal to x.

The class **Oz** includes **On**, the class of ordinal numbers. In addition, the omnific integers are closed under addition, subtraction, and multiplication; **Oz** is therefore a subRing of **NO**, with **NO** being the field of fractions of **Oz**. This idea is encompassed in the following theorem.

Theorem 3.3 (Conway)**.** *Every surreal number x can be represented as a quotient of two omnific integers.*

Proof. Conway showed in [2] that every surreal number x can be expressed in its normal form as

$$x = \sum_{\mathcal{B} < \mathcal{A}} \omega^{y_{\mathcal{B}}} \times r_{\mathcal{B}},$$

where $\mathcal{A}$ is an ordinal, each $r_{\mathcal{B}}$ is a non-zero real, and the numbers $y_{\mathcal{B}}$ form a descending sequence of numbers. In the case of omnific integers, the form is modified to $x = \sum \omega^y \cdot r_y$, where $r_y = 0$ for $y < 0$ and $r_0 \in \mathbb{Z}$. Thus, with $r_y = 0$ for $y \leq -\mathcal{A}$ (where $\mathcal{A}$ is some ordinal), we have that both $\omega^{\mathcal{A}}$ and $x\omega^{\mathcal{A}}$ are integers; this means that we can write any surreal number x as the quotient of the integers $\omega^{\mathcal{A}}$ and $x\omega^{\mathcal{A}}$. ■

3.1. **Continued Fractions.** For any positive number x, we know that $[x] \leq x < [x] + 1$, where $[x]$ is the integer part of x. Let $a = [x]$, such that if $x \neq a$, then we can write $x = a + \frac{1}{y}$. Furthermore, if we have $y \neq [y]$, then we can write y as $b + \frac{1}{z}$. Note that if none of the remainders $(y, z, \ldots)$ are 0, then we obtain the infinite continued fraction

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cdots}}}$$

If one of the remainders does end up being 0, then the continued fraction terminates, and we call the number a *fractional number*. In this case, the numbers $x + 1$, $\frac{1}{x}$, and $-x$ are all fractional numbers as well.

Consider the equation $x^2 - Ny^2 = \pm 1$, where $N \in \mathbb{Z}$ and $x, y \in$ **Oz**. Of this equation (known as **Pell's equation**), Conway claimed in [2] that $\frac{x}{y}$ must be one of the convergents to the continued fraction of $\sqrt{N}$. Consider the case of $N = \omega + 3$. In this case, we find that $\sqrt{\omega+3} = \sqrt{\omega} + \frac{1}{u}$, for some u; in addition,

$$u = \frac{1}{3}\left(\sqrt{\omega+3} + \sqrt{\omega}\right) = \frac{2}{3}\sqrt{\omega} + \frac{1}{v},$$

for some v such that

$$v = \sqrt{\omega+3} + \sqrt{\omega} = 2\sqrt{\omega} + \frac{1}{u}.$$

We thus find that the continued fraction for $\sqrt{\omega+3}$ is

$$\sqrt{\omega+3} = \sqrt{\omega} + \cfrac{1}{\frac{2}{3}\sqrt{\omega} + \cfrac{1}{2\sqrt{\omega} + \cfrac{1}{\frac{2}{3}\sqrt{\omega} + \cdots}}}$$

We see that of the convergents of this continued fraction, namely

$$\frac{\sqrt{\omega}}{1}, \qquad \frac{\frac{2}{3}\omega + 1}{\frac{2}{3}\sqrt{\omega}}, \qquad \frac{\frac{4}{3}\omega\sqrt{\omega} + 3\sqrt{\omega}}{\frac{4}{3}\omega + 1}, \qquad \frac{\frac{8}{9}\omega^2 + \frac{8}{9}\omega + 1}{\frac{8}{9}\omega\sqrt{\omega} + \frac{4}{3}\sqrt{\omega}}, \qquad \ldots,$$

the second, fourth, sixth, and in general the alternating convergents provide solutions to Pell's equation. For example,

$$\left(\frac{2}{3}\omega + 1\right)^2 - (\omega+3)\left(\frac{2}{3}\sqrt{\omega}\right)^2 = 1.$$

4. Pseudo–Numbers and Games

We will now investigate what happens if we drop the requirement that surreal numbers must be well-formed (that is eve4ry number corresponds to two sets of previously created numbers, such that no member of the left set is greater than or equal to any member of the right set). Knuth [3] called numbers that are not well-formed as PSEUDO-NUMBERS. For an example $\{0 \mid 0\}$ is a pseudo–number since $0 \geqslant 0$. Investigation of pseudo-numbers leads to some interesting results. Obviously, any theorem or property that we have proved using the well-formedness of surreal numbers may not be true for pseudo-numbers.

Now consider the pseudo-number $\{1 \mid 0\}$. We see that $\{1 \mid 0\} \nleqslant 0$, since $1 > 0$ and $0 \nleqslant \{1 \mid 0\}$, since $0 \leqslant 0$. Thus we can say that 0 and $\{1 \mid 0\}$ are not related at all. Here we see that unlike numbers, pseudo-numbers are not completely ordered.

Although pseudo-numbers do not behave quite as nicely as numbers, we can still verify certain properties (such as transitivity) for them. As the numbers go, pseudo–numbers are obviously not very useful, so it makes sense to require that surreal numbers be well–formed. However, pseudo–numbers play a useful role in game theory.

We will not here dig much into the application of surreal numbers and pseudo–numbers to game theory; but a few things are worth noting.

We will consider games played by two players called LEFT and RIGHT The games involve no luck and no hidden information. Chess is an example of such a game.

Let x be a position in a game. If LEFT is to move, he can turn the position x into a number of other positions, x_1, x_2, x_3. If RIGHT is to move, she can turn the position x into a number of other positions say, z_1, z_2, z_3. We will write this thus:

$$x = \{x_1, x_2, x_3 \mid z_1, z_2, z_3\}$$

It's not hard to see that here, x is written as a surreal number or a pseudo–number whose left set consists of the positions that can be reached if LEFT is to move, and whose right set consists of the positions that can be reached if RIGHT is to move.

If the next player to move finds that he has lost, he has no moves to make. So, for example, a number whose right set is empty denotes a position where RIGHT has lost if she is the player that should make the next move. We can now make a few observations:

(1) $0 = \{|\}$ is a position where the next player to move has lost, that is $0 \in \mathcal{N}$.
(2) $1 = \{| \, 0\}$ is a position which Left will win either because RIGHT is about to move but has no legal moves left, or because LEFT is about to move and creates the position 0 (which is an $\mathcal{N}$ position as seen before), in which RIGHT has lost.
(3) $-1 = \{0 \, |\}$ is a position in which Right will win, that is $-1 \in \mathcal{R}$.
(4) $\{0 \mid 0\}$ is a position where the next player to move will win, because the move will lead to a position where the next player to move has lost. Thus $\{0 \mid 0\} \in \mathcal{N}$.

Conway in [1] calls $\{0 \mid 0\}$ *star* and denotes it by the symbol $*$. The pseudo-number $\{* \mid *\}$ is a position where the next player to move will lose, because the move will lead to position $\{0 \mid 0\}$, in which the next player to move will win. Thus we deduce that both $\{* \mid *\}$ and $\{|\}$ identify a position in which the next player to move will lose. The amazing thing now is that if we compare $\{* \mid *\}$ and $\{|\}$ using our well known definitions, we get $\{* \mid *\} = \{|\}$.

5. THE FIELD $\mathbf{On}_2$

What we get is in sense the characteristic 2 analogue of the large field **NO** (which we saw in section 2), which we might call $\mathbf{NO}_2$ naturally. But it turns out that this new field is also the simplest way of turning the class **On** of all the ordinals into a Field. And so for the moment we will adopt the name $\mathbf{On}_2$ (which has in any case a nicer sound).

The next thing to ask would be, how can we find a simplest addition and multiplication which would make **On** a Field? We leave this as an exercise to the reader (hint: think about the NIM–addition and NIM–product). For notational convenience, let α, β be ordinals and α', β' represent arbitrary ordinals smaller than α, β. We shall write $\operatorname{mex}(S)$ for the least ordinal not in set S, and refer to the members of S as excludents. If $\alpha = \operatorname{mex}(S)$, then we shall often use $\alpha*$ for the variable ranging over the set S. More precisely $\alpha*$ may take all the values less than α and possibly some values greater than α, but not α itself.

5.1. Properties of Addition and Multiplication.

Let's look at addition properties first.

Proposition 5.1. *We have $\alpha + \beta = \alpha + \gamma$ if an only if $\beta = \gamma$, moreover*

$$\alpha + \beta = mex\{\alpha * + \beta, \alpha + \beta *\}$$

Proof. On a contrary say, without loss of generality $\beta > \gamma$, then $\alpha + \gamma$ is an excludent for $\alpha + \beta$. The second part follows, for certainly all numbers $\alpha' + \beta, \alpha + \beta'$ are excludents, and the other excludents are distinct from $\alpha + \beta$. This completes the proof. ■

Proposition 5.2. *For all ordinals α, β, γ we have*

(1) $\alpha + 0 = \alpha$
(2) $\alpha + \beta = \beta + \alpha$
(3) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$
(4) $\alpha + \alpha = 0$
(5) $\alpha = -\alpha$

Proof. These claims have 1–line proofs as follows

(1) $\alpha + 0 = \text{mex}\{\alpha' + 0, \alpha + 0'\} = \text{mex}\{\alpha'\} = \alpha.$
(2) $\alpha + \beta = \text{mex}\{\alpha' + \beta, \alpha + \beta'\} = \text{mex}\{\beta + \alpha', \beta' + \alpha\} = \beta + \alpha.$
(3) The proof is as follows

$$\begin{aligned}(\alpha + \beta) + \gamma &= \text{mex}\{(\alpha + \beta) * + \gamma, (\alpha + \beta) + \gamma'\} \\ &= \text{mex}\{(\alpha' + \beta) + \gamma, (\alpha + \beta') + \gamma, (\alpha + \beta) + \gamma'\} \\ &= \text{mex}\{\alpha' + (\beta + \gamma), \alpha + (\beta' + \gamma) + \gamma, \alpha + (\beta + \gamma')\} \\ &= \cdots = \alpha + (\beta + \gamma).\end{aligned}$$

(4) $\alpha + \alpha = \text{mex}\{\alpha' + \alpha, \alpha + \alpha'\} = \text{mex}\{0*\} = 0.$
(5) $-\alpha = \text{mex}\{-\alpha'\} = \text{mex}\{\alpha'\} = \alpha.$

as desired. This completes the proof of proposition. ■

Hence $\mathbf{On}_2$ forms a commutative Group with 0 for zero and $-\alpha = \alpha$. Now since we have seen properties of ordinals under addition, lets see how they behave under multiplication.

Proposition 5.3. *For all ordinals α, β, γ we have*

(1) $\alpha 0 = 0.$
(2) $\alpha 1 = \alpha.$
(3) $\alpha\beta = \beta\alpha.$
(4) $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma.$
(5) $(\alpha\beta)\gamma = \alpha(\beta\gamma).$

Proof. These also have 1–line proofs as follows:

(1) $\alpha 0 = \text{mex}\{\} = 0.$
(2) $\alpha 1 = \text{mex}\{\alpha' 1 + \alpha 0 - \alpha' 0\} = \text{mex}\{\alpha'\} = \alpha.$
(3) $\alpha\beta = \text{mex}\{\alpha'\beta, \alpha' a, \alpha'\beta'\} = \text{mex}\{\alpha\beta', \alpha'\beta, \alpha'\beta'\} = \beta\alpha$
(4) $(\alpha + \beta)\gamma = \text{mex}\{(\alpha + \beta) * \gamma + (\alpha + \beta)\gamma' - (\alpha + \beta) * \gamma'\}$

$$\begin{aligned}&= \text{mex}\{(\alpha' + \beta)\gamma + (\alpha + \beta)\gamma' - (\alpha' + \beta)\gamma', (\alpha + \beta')\gamma + (\alpha + \beta)\gamma' - (\alpha + \beta')\gamma'\} \\ &= \text{mex}\{\beta\gamma + (\alpha'\gamma + \alpha\gamma' - \alpha'\gamma'), \alpha\gamma + (\beta'\gamma + \beta\gamma' - \beta'\gamma')\} \\ &= \text{mex}\{(\alpha\gamma) * + \beta\gamma, \beta\gamma(\beta\gamma) * +\} = \alpha\gamma + \beta\gamma\end{aligned}$$

(5)

$$\begin{aligned}(\alpha\beta)\gamma &= \text{mex}\{(\alpha\beta) * \gamma + (\alpha\beta)\gamma' - (\alpha\beta) * \gamma'\} \\ &= \text{mex}\{(\alpha'\beta + \alpha\beta' - \alpha'\beta')\gamma + (\alpha\beta)\gamma' - (\alpha'\beta + \alpha\beta' - \alpha'\beta')\gamma'\} \\ &= \text{mex}\{(\alpha'\beta\gamma + \alpha\beta'\gamma - \alpha'\beta'\gamma' + \alpha'\beta'\gamma - \alpha'\beta\gamma' + \alpha\beta'\gamma' - \alpha'\beta'\gamma')\} \\ &= \cdots = \alpha(\beta\gamma).\end{aligned}$$

where the last two equalities have followed simply because

$$\alpha\beta = \text{mex}\{\alpha * \beta + \alpha\beta * - \alpha * \beta *\}$$

■

Thus we see that $\mathbf{On}_2$ is a commutative Ring with 1 as one. In fact $\mathbf{On}_2$ is a Field, for we can use the analogue of our genetic construction inverses (see [2]) in **NO** to construct the inverse in $\mathbf{On}_2$. The results we will now prove show that each new number extends the set of previous ones in the simplest possible way, regarding as addition simpler than multiplication and division, and these as simpler than algebraic extensions which are in turn simpler than the transcendental ones. In stating our results we will follow Von Neumann's convention of identifying each ordinal number with the set of all previous ones. So when we say, for instance, that 6 is a field, we mean that the set $\{0, 1, 2, 3, 4, 5\}$ is a field.

We shall use [square brackets] for the ordinal operations, for instance we have $[4 + 4] = 8$, $[4 \cdot 4] = 16, [4^4] = 256$. We will use Δ as a name for some ordinal whose arithmetic relation to earlier ordinals is been considered, and δ for ordinals less than Δ.

Proposition 5.4. *If Δ is not a group (under addition), then $\Delta = \alpha + \beta$, where (α, β) is any lexicographically earliest pair of numbers in Δ whose sum is not in Δ.*

The proof turns out to be pretty trivial since $\alpha + \beta \geqslant \Delta$. But the excludents $\alpha' + \beta, \alpha + \beta'$ for $\alpha + \beta$ are all in Δ and hence $\alpha + \beta \geqslant \Delta$, as desired. ■

Proposition 5.5. *If Δ is a group, we have $[\Delta\alpha] + \beta = [\Delta\alpha + \beta]$, for all $\alpha, \beta \in \Delta$.*

Proof. In this case, the excludents are $[\Delta\alpha' + \delta] + \beta$ and $[\Delta\alpha] + \beta'$. But since we already know that Δ forms a group, we can solve the equation $\delta + \beta = \overline{\delta}$ for any given $\overline{\delta} \in \Delta$, and so by induction we deduce that the excludents are

$$[\Delta\alpha'] + \delta + \beta = [\Delta\alpha'] + \overline{\delta}, \text{ and } [\Delta\alpha + \beta']$$

which are precisely numbers less than $[\Delta\alpha + \beta]$, as desired. This completes the proof. ■

Proposition 5.6. *If Δ is a group but not a ring, then we have $\Delta = \alpha\beta$ where (α, β) is any lexographically earliest pair of numbers in Δ whose product is not in Δ.*

The proof is left as an easy exercise to the reader. Conway in [2] proves that If Δ is a field but not algebraically closed, then Δ is a root of the lexicographically earliest polynomial having no root in Δ. Therefore we can deduce that each ordinal Δ extends the set of all positive ordinals in the simplest possible way.

References

[1] Norman Larrabee Alling. *Foundations of Analysis over Surreal Number Fields, Volume 141*. North-Holland Publishing Company, 1987.

[2] John H Conway. *On Numbers and Games*. CRC Press, 2000.

[3] Donald Ervin Knuth. *Surreal numbers: How Two Ex-students Turned on to Pure Mathematics and Found Total Happiness : a Mathematical Novelette*. Addison-Wesley Professional, 1974.

[4] Simon Rubinstein-Salzedo and Ashvin Swaminathan. Analysis on surreal numbers. *arXiv preprint arXiv:1307.7392*, 2013.

[5] Simon Rubinstein-Salzedo and Ashvin Swaminathan. Surreal analysis: An analogue of real analysis for surreal numbers. *Journal of Logic and Analysis*, page 1–39, 2014.

[6] Thea van Roode. Partizan forms of hackenbush combinatorial games, 2002.