THE GAME OF KONANE

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1. Preliminaries

1.1. **The Rules of Konane.** The game of KONANE starts with a checkered board, filled in with black and white stones such that no two stones of the same color occupy adjacent squares. Two adjacent pieces of the board are then removed. A player moves by jumping one of eir stones (black for Left, and white for Right) over a different colored stone orthogonally. The jumped stone is then removed. A player may make multiple jumps in one turn, but may not change direction mid-turn. However, ey need not make multiple jumps. As usual, the winner is the player who makes the last move. In the generalized version of KONANE, two stones of the same color may occupy adjacent cells.

1.2. **Habitats.**

Definition 1.1. Let $\mathscr{S}$ be a set of combinatorial game values. A ruleset A is a habitat of $\mathscr{S}$ if, for every $G \in \mathscr{S}$, there is a position of A with game value equal to G.

A clear example of this is that if $\mathscr{S} = \{*0, *1, *2, *3, \ldots\}$ is the set of nimbers, NIM is a habitat of $\mathscr{S}$. However, in general, not every value arising from A needs to be in $\mathscr{S}$. For example, NIM is a habitat of $\{*\}$, and HACKENBUSH is a habitat of $\{1\}$, though both of these rulesets take on more values.

Recall that a game is short if it is born before day ω.

Definition 1.2. The short Conway group is the group containing the short Normal-play games.

Definition 1.3. A ruleset A is universal if it is a habitat of the short Conway group.

2. Specific Konane Positions

2.1. **Linear Patterns.** Let $\mathrm{L}(n)$ denote an alternating row of stones, starting with a black stone. For example,

$$\mathrm{L}(5) = \bullet \circ \bullet \circ \bullet.$$

Let us investigate how to express the value of $\mathrm{L}(n)$. Clearly, $\mathrm{L}(0) = \mathrm{L}(1) = 0$. It is also clear that $\mathrm{L}(2) = \{0 \mid 0\} = *$. Consider $\mathrm{L}(3) = \bullet \circ \bullet$. Left has no moves, while Right can move to 0 in two ways. Thus, this is $\{|0\} = -1$. The position $\mathrm{L}(4)$ is 0 by inspection, while $\mathrm{L}(5)$ is -2: Left has no moves, and both of Right's moves go to $L(3) = -1$. It is not hard to see that the following table is correct.

n	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{L}(n)$	0	0	$*$	-1	0	-2	$*$	-3	0	-4	$*$

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With this, we get the pattern that

$$\begin{aligned} \mathrm{L}(2j+1) &= -j \\ \mathrm{L}(4j) &= 0 \\ \mathrm{L}(4j+2) &= *, \end{aligned}$$

which we prove by induction on j. The table shows that this holds for the base case. Now, notice that every jump changes $L(n)$ to $L(n-2)$, because the jumping stone is stranded away from the main game, and so effectively removed from play. If n is even, either player can move, but if n is odd, only Right can move. Thus,

$$\begin{aligned} \mathrm{L}(2j+1) &= \{\,|\,\mathrm{L}(2j-1)\} = \{\,|-(j-1)\} = -j \\ \mathrm{L}(4j) &= \{\mathrm{L}(4j-2)\,|\,\mathrm{L}(4j-2)\} = \{*|*\} = 0 \\ \mathrm{L}(4j+2) &= \{\mathrm{L}(4j)\,|\,\mathrm{L}(4j)\} = \{0|\} = *. \end{aligned}$$

We can also consider adding a stone diagonally. We denote this by $\mathrm{LOT}_1(n)$ (linear with offset tail), where there are n stones in total. For example

$$\mathrm{LOT}_1(3) = \circ^{\circ\bullet}.$$

Note that we now start with a white stone.

We can once again characterize these.

$$\begin{aligned} \mathrm{LOT}_1(3) &= \downarrow \\ \mathrm{LOT}_1(2j) &= j-1 \\ \mathrm{LOT}_1(4j+1) &= * \\ \mathrm{LOT}_1(4j+3) &= 0. \end{aligned}$$

It is easy to see that $\mathrm{LOT}_1(3) = \{* \mid 0\} = \downarrow$. Now, $\mathrm{LOT}_1(2) = 0$, and

$$\mathrm{LOT}_1(2j) = \{\mathrm{LOT}_1(2j-2), * - \mathrm{L}(2j-3) \mid\} = \{* + j - 2, j - 2 \mid\}.$$

Subtracting $j-1$ from this, it is easy to see that the resulting game is in $\mathcal{P}$. We also have

$$\begin{aligned} \mathrm{LOT}_1(4j+1) &= \{* - \mathrm{L}(4j-2)\,|\,\mathrm{LOT}_1(4j-1)\} = \{0\,|\,\mathrm{LOT}_1(4j-1)\} = * \\ \mathrm{LOT}_1(4j+3) &= \{* - \mathrm{L}(4j)\,|\,\mathrm{LOT}_1(4j+1)\} = \{*\,|\,\mathrm{LOT}_1(4j+1)\} = 0. \end{aligned}$$

2.2. **Rubber Bands.** We will use coordinates to identify where the stones are in a KONANE position. For example, $\mathrm{L}(n)$ has black stones at $(0,0),(2,0),\ldots$ and white stones at $(1,0),(3,0),\ldots$. If P is a position with a stone at (a,b), we write $P \setminus (a,b)$ for the position without a stone at (a,b).

Lemma 2.1. *Consider the KONANE pattern shown in Figure 1. Denote this by P_n, so P_n is a $(2n+3)\times 5$ rectangle, where the black pieces occupy $(1,2),(2,2),(1,4),(2,4),\ldots,(1,2n),(2,2n)$ and the white pieces occupy $(1,0)$, $(1,2n+2)$, $(1,1),(1,3),\ldots,(1,2n+1)$, and $(3,2),(4,2),(3,4),(4,4),\ldots,(3,2n),(4,2n)$. Write $a(k)$ for the number of black stones above the line $2k+1$, and $b(k)$ for the number of black stones below $2k+1$.*

1. $P_n = 0$.
2. $P_2 \setminus (1,3) = -1$.
3. $P_n \setminus (1,2k+1) = -\max(a(k),b(k))$.

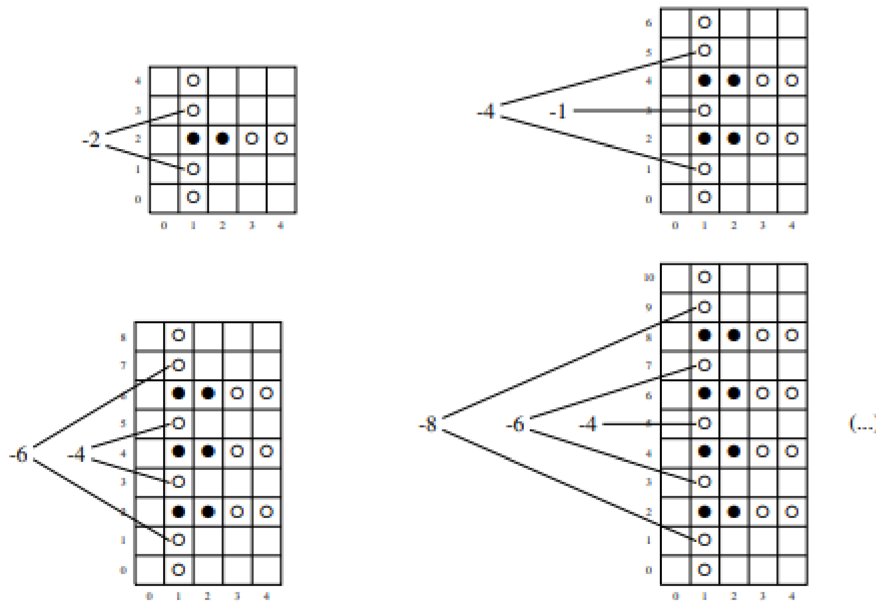

Figure 1. Rubber bands in KONANE.

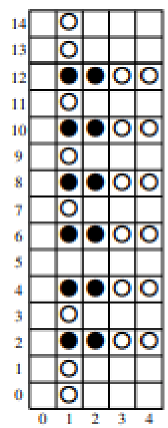

Figure 2. One case of a rubber band.

Before proving this, we consider the case in Figure 2.

If Right moves the stone from $(1,7)$ to $(1,5)$, we get the game $\{-8 \mid -6\} = -7$, while if he moves the stone at $(1,3)$, we get $\{-4 \mid -2\} = -3$. Thus, the position is $\{\mid -7, -3\} = \{\mid -7\} = -8$. In this case, Right's move is made in the opposite direction of the largest number of black stones.

Proof of Lemma 2.1. Both (1) and (2) are clear by inspection. As in the previous example, Right's move must be made in the opposite direction of the largest number of black stones. This gives the game $\{\mid -\max(a(k), b(k)) + 1\} = -\max(a(k), b(k))$. ∎

2.3. **Taps.**

Lemma 2.2. *Consider the* KONANE *pattern in Figure 3. The position* P_n *is a* $7\times(2n+3)$ *rectangle, with black pieces in* $(2n+2, 2)$, $(2n+3, 3)$ *and* $(2,4), (3,2), (4,4), (5,2), \ldots, (2n,4), (2n+$

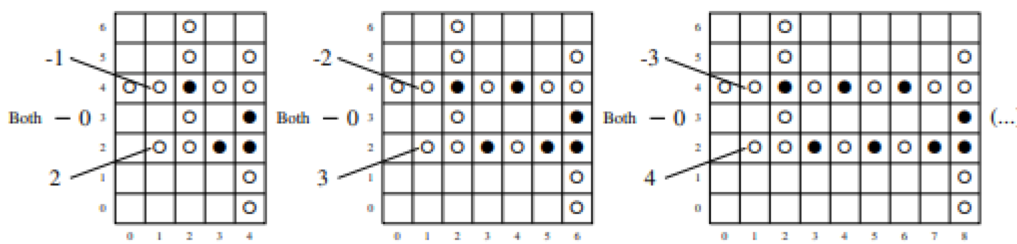

Figure 3. Taps in KONANE

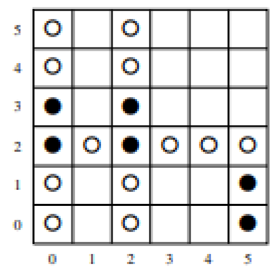

Figure 4. Joining of rubber bands.

$1, 2)$*, and white pieces in* $(0, 4)$, $(1, 2)$, $(1, 4)$, $(2, 2)$, $(2, 3)$, $(2, 5)$, $(2, 6)$, $(2n+2, 0)$, $(2n+2, 1)$, $(2n + 2, 4)$, $(2n + 2, 5)$, *and* $(3, 4), (4, 2), (5, 4), \ldots, (2n, 2), (2n + 1, 4)$.

(1) $P_n = 0$.
(2) $P_n \setminus (1, 2) = n + 1$.
(3) $P_n \setminus (1, 4) = -n$.
(4) $P_n \setminus \{(1, 2), (1, 4)\} = 0$.

Proof. (1) is clearly true. In $P_n \setminus (1, 4)$, Right can take, one by one, all n black pieces in line 4, so the value is $-n$. Similarly, in $P_n \setminus (1, 2)$, Left can take, one by one, all n white pieces in line 2, along with that in $(2, 3)$, so this is $n + 1$. Finally, in $P_n \setminus \{(1, 2), (1, 4)\}$, there are exactly n moves for both players, so the first player loses. ■

3. Konane is Universal

We now prove that KONANE is universal. Our proof of this fact is constructive. The games 0, 1 and $*$ are

●, ● ● ○, ●○

respectively. We can also join rubber bands without changing values, as in Figure 4.

We now present the connecting scheme, as shown in Figure 5. If a black stone moves to $(5, 1)$, removing the stones at $(0, 1)$, $(2, 1)$ and $(4, 1)$ in the process, the game value becomes 1. Moreover, we can choose rubber band sizes such that the incomplete captures are reversible options (this follows from the fact that the largest game born on day n is n, and Lemma 2.1, and will be explained in more detail later).

We also have the general idea of the recursion in Figure 6. In the initial position, the focal point consists of the pair of cells occupied by the only pair of cells that can move.

The construction of each rectangular position is made with rubber bands, turning points (based in taps), one link (which is a rubber band), and the rectangular positions of the

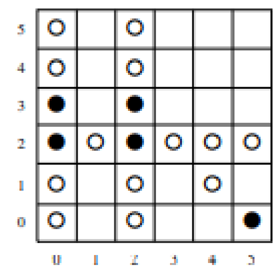

Figure 5. An example of the connecting scheme.

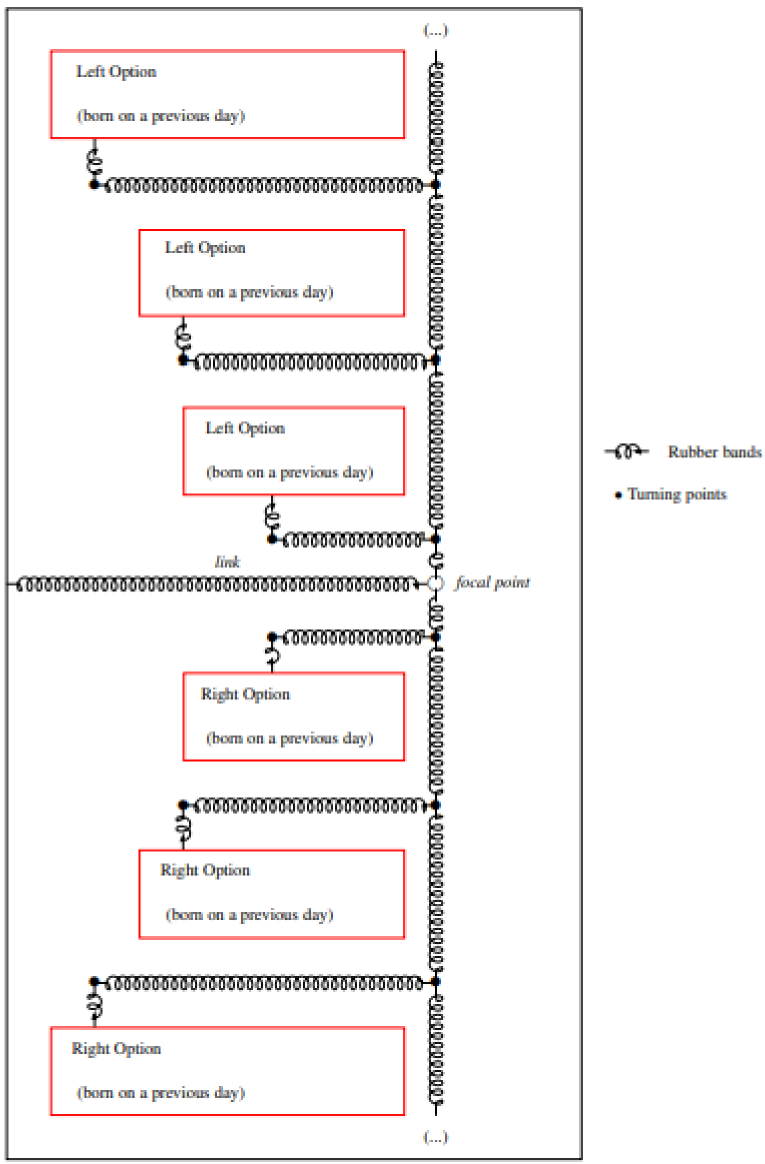

Figure 6. An example of recursion.

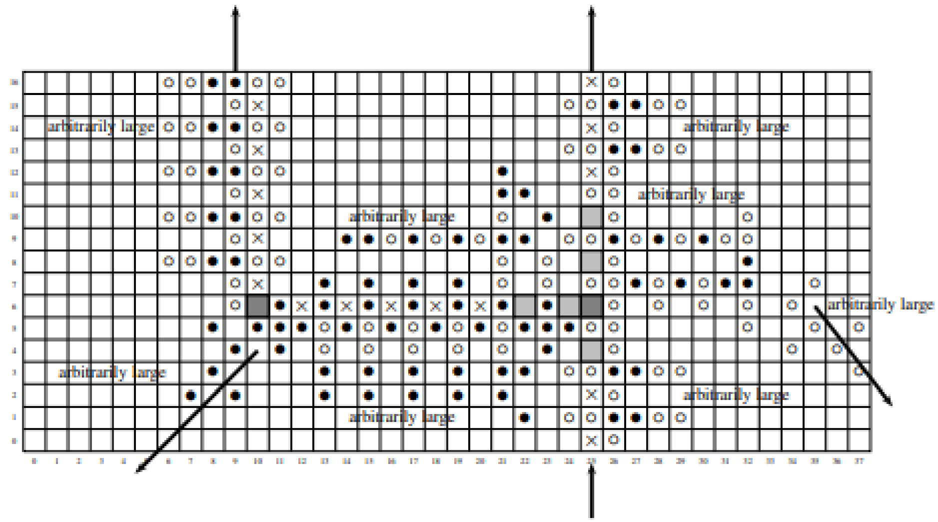

Figure 7. An example of turning points.

previous days. Note that there are no moves in these rectangular positions. The focal point is occupied by opposing stones only after a complete capture through the link has occurred. After this, a game value of the previous days is obtained.

When we construct a game of day n with options of the previous days, we choose rubber band sizes in such a way so that, for every incomplete Left (Right) capture, there is an integer $G^{LR} \leq -n$ ($G^{RL} \geq n$). Then, these options reverse out.

We now describe how to construct turning points. When Left (Right) captures to a turning point, she (he) makes a threat larger than or equal to n (smaller than or equal to $-n$). To defend the threat, Right (Left) has only one good option, which creates a threat smaller than or equal to $-n$ (larger than or equal to n). After these two forced moves, the only way for Left (Right) to defend the second threat is to choose an option of the previous days.

For example, consider the turning point shown in Figure 7. The crosses represent options that are adjacent to rubber bands and reverse out, so when we have a sequence of captures made by a black stone in column 25, we only need to look at the moves to the gray and dark gray cells. The turning point is $(25, 6)$.

- If Left captures to $(25, 4)$, Right captures to the call $(22, 4)$, obtaining greater than or equal to $-n$ points, so this move reverses out.
- If Left captures to $(25, 8)$, Right captures to $(25, 7)$, once again obtaining more than or equal to $-n$ points (the tap is arbitrarily large), so this move once again reverses out.
- If Left captures to $(25, 10)$, she completely opens the tap, turning it to 0. Right responds by capturing to $(22, 10)$, and afterwards, Right's stone in $(22, 10)$ will capture to $(22, 8)$, obtaining at least $-n$ points.

After Left moves to $(25, 6)$, she creates a threat in $(35, 6)$ to get at least n points. If Right captures to $(24, 6)$, Left replies by capturing to $(24, 7)$. The next move from $(24, 7)$ to $(20, 7)$ will get at least n points, so this move of Right's reverses out. If he instead captures to $(22, 6)$, Left captures to $(22, 7)$, obtaining at least n points, so this move by Right once again

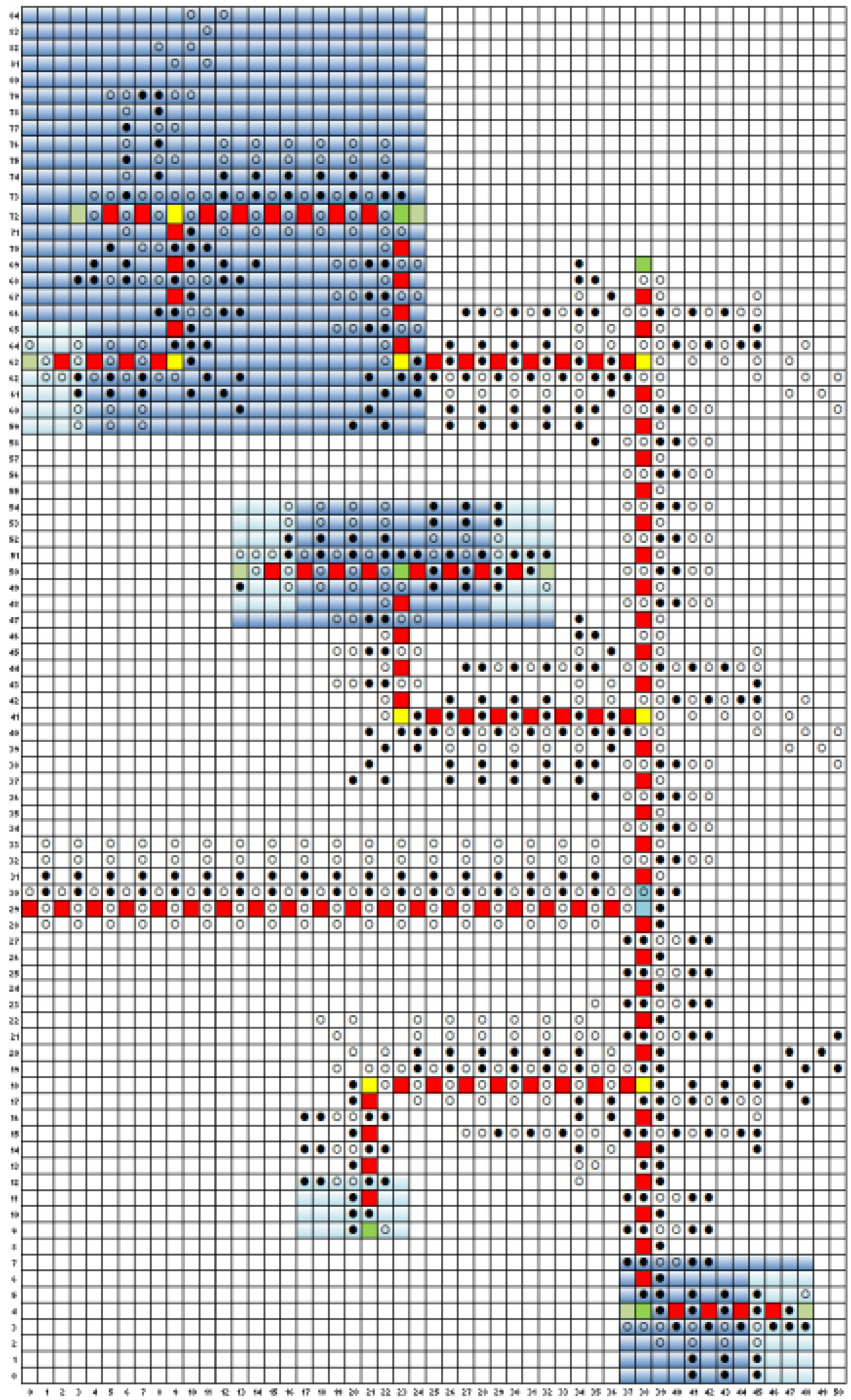

Figure 8. A game made using the recursion in KONANE. The colored rectangles are the options of the previous days.

reverses out. Thus, the good move is to $(10, 4)$, creating a threat of at least $-n$ points. Left has to go to an option that is a game of the previous days.