WINNING STRATEGIES FOR WYTHOFF'S GAME

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1. Introduction

We can split combinatorial games into two categories, partisan and impartial. This paper will focus on impartial games, where both players have the same moves available to them. Perhaps the most famous impartial game is the game of NIM, which consists of some piles of rocks. Each turn, a player may take as many rocks as he or she wants, but only from a single pile. The player who is unable to make a move loses the game.

WYTHOFF'S GAME has similar rules to NIM, but there are typically only 2 piles at most. In addition to the moves allowed by NIM, players may also take stones from both piles on the same turn, provided an equal amount is taken from each pile. WYTHOFF'S GAME positions can be represented as (a, b), where a and b are the sizes of the two piles. Additionally, since the order of the piles does not matter, we will always write positions such that $a \leq b$ for convenience.

Another way to visualize WYTHOFF'S GAME is as a chess queen on an arbitrarily large chessboard, where both players take turns moving the queen with a few caveats. The queen must always move left, down, or diagonally in those two directions, and the winning condition is the same: the player unable to move loses, which only happens when the other player moves the queen into the bottom left corner. Then we can give coordinates (a, b) to each square on the chessboard, starting with $(0, 0)$ in the bottom left corner. The game with the queen starting on (a, b) is then analogous to a WYTHOFF'S GAME position (a, b).

In this paper, we will explore the winning strategy for the basic 2-pile WYTHOFF'S GAME, as well as more complicated positions with more piles.

2. Winning and Losing Positions

In impartial games, we split positions into two categories: winning positions and losing positions, or $\mathcal{N}$ and $\mathcal{P}$ positions, respectively.

Definition 1. Assuming both players play optimally, we call a position an $\mathcal{N}$ position if the next player to move can guarantee a win, and we call a position a $\mathcal{P}$ position if the next player to move will always lose.

Every position is either an $\mathcal{N}$ or a $\mathcal{P}$ position, and they build recursively off of each other.

Theorem 2. *Every $\mathcal{N}$ position has some move which results in a $\mathcal{P}$ position, while every move from a $\mathcal{P}$ position results in an $\mathcal{N}$ position.*

This theorem is easy to prove because it makes logical sense: a winning position allows you to give your opponent a losing one, and a losing position forces you to give your opponent a winning position. The above theorem simply formalizes these intuitive definitions. In WYTHOFF'S NIM and many other combinatorial games, the most basic $\mathcal{P}$ position is the position where there are no available moves, which in this case is $(0, 0)$. We will be focusing on $\mathcal{P}$ positions and how to easily find them for two reasons: knowing the $\mathcal{P}$ positions allows optimal play, and $\mathcal{P}$ positions are difficult to recursively find since every possible move must be checked.

$\mathcal{N}$ positions are comparatively very easy to find, since we only need to find a winning move. For example, we know that all positions of the form $(0, a)$ or (a, a) are $\mathcal{N}$ positions because we can reach $(0, 0)$ in one move from those positions. After $(0, 0)$, the next $\mathcal{P}$ position is $(1, 2)$: the available moves are to $(0, 1)$, $(0, 2)$, and $(1, 1)$, which are all $\mathcal{N}$ positions since we can move to $(0, 0)$ in one move. If we continue to use the theorem concerning $\mathcal{N}$ and $\mathcal{P}$ positions to exhaustively check for more $\mathcal{P}$ positions, we can eventually make a table of the first few:

P_n	0	1	2	3	4	5	6	7	8	9
a_n	0	1	3	4	6	8	9	11	12	14
b_n	0	2	5	7	10	13	15	18	20	23

Using the table above and our theorem concerning the relationship between $\mathcal{N}$ and $\mathcal{P}$, we can piece together a strategy to play well in various positions of WYTHOFF'S GAME that have relatively few stones. The basic strategy to win in $\mathcal{N}$ positions is to just move to $\mathcal{P}$ positions, which we can see in action in the following example.

Example 3. We can give a winning strategy for a position such as $(5, 7)$ using what we know so far. From $(5, 7)$, we can move to two different $\mathcal{P}$ positions, $(3, 5)$ and $(4, 7)$, but moving to $(3, 5)$ ends the game faster. From $(3, 5)$, the second player has many options: $(0, 2), (1, 3), (2, 4)$, $(0, 5), (1, 5), (2, 5), (3, 4), (3, 3), (2, 3)$, and $(0, 3)$. We respond to $(0, 2)$,

$(0,5), (0,3)$, and $(3,3)$ with $(0,0)$, and respond to all the other moves with $(1,2)$, which we already saw the winning strategy for. Thus, our strategy works to guarantee a win in the position $(5,7)$.

We have the basics of playing WYTHOFF'S GAME well, but to play in positions with many stones we need a fast way to determine the losing positions.

3. How to Find Losing Positions

It turns out there is a simple explicit formula for $\mathcal{P}$ positions outlined by Wythoff himself in his paper on the subject.

Theorem 4. *The $\mathcal{P}$ positions of* WYTHOFF'S GAME *are those of the form $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$ for values of $n \geq 0$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\lfloor s \rfloor$ is equal to s rounded down to the nearest integer* **[Wyt07]**.

We can check that the values in the table line up with the values predicted by our new theorem.

Example 5. $(\lfloor 0\phi \rfloor, \lfloor 0\phi^2 \rfloor) = (\lfloor 0 \rfloor, \lfloor 0 \rfloor) = (0,0)$,
$(\lfloor \phi \rfloor, \lfloor \phi^2 \rfloor) = (\lfloor 1.618\ldots \rfloor, \lfloor 2.618\ldots \rfloor) = (1,2)$,
$(\lfloor 2\phi \rfloor, \lfloor 2\phi^2 \rfloor) = (\lfloor 3.236\ldots \rfloor, \lfloor 5.236\ldots \rfloor) = (3,5)$,
$(\lfloor 3\phi \rfloor, \lfloor 3\phi^2 \rfloor) = (\lfloor 4.854\ldots \rfloor, \lfloor 7.854\ldots \rfloor) = (4,7)$,
$(\lfloor 4\phi \rfloor, \lfloor 4\phi^2 \rfloor) = (\lfloor 6.472\ldots \rfloor, \lfloor 10.472\ldots \rfloor) = (6,10)$. Indeed, these values for the first five $\mathcal{P}$ positions match with those in the table.

This theorem also helps explain a few patterns which show up in the table above. We have $\phi^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} = 1 + \phi$, which means that we can substitute for ϕ^2 to get a different form for the $\mathcal{P}$ positions: $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) = (\lfloor \phi n \rfloor, \lfloor (\phi+1)n \rfloor) = (\lfloor \phi n \rfloor, \lfloor \phi n \rfloor + n)$. Looking at the table, we can indeed see that $b_n = a_n + n$ does hold within the table. Our calculation shows that this remains true for triples (a_n, b_n, n) beyond the table. Another pattern we may notice is that no positive integer shows up in the table more than once. It makes sense in the context of WYTHOFF'S GAME why two $\mathcal{P}$ positions can't share a number since we can't move from a $\mathcal{P}$ position to another. However, it is not immediately obvious why this should result from our explicit formula.

Interestingly, our formula has connections with results in number theory that explain our previous observation. The following is often known as *Beatty's theorem*:

Theorem 6. *Given two irrational numbers (r,s) such that $r, s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, the sequences $\lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, \ldots$ and $\lfloor s \rfloor, \lfloor 2s \rfloor, \lfloor 3s \rfloor, \ldots$ contain all positive integers exactly once between them.*

Two proofs of this theorem can be found here: **[Bea+27]**. We can quickly check that ϕ and ϕ^2 satisfy the conditions for (r, s): they are both irrationals greater than 1, and dividing both sides of $\phi^2 = \phi + 1$ by ϕ^2 gives $1 = \frac{1}{\phi} + \frac{1}{\phi^2}$. This explains why the non-$(0, 0)$ $\mathcal{P}$ positions of WYTHOFF'S GAME contain every positive integer exactly once, and could be useful for proving Theorem 4. We can also use the results of this theorem to give a better recursive formula for $\mathcal{P}$ positions:

Corollary 7. *We define* $\operatorname{mex}(a_1, a_2, \ldots, a_n)$ *as the smallest nonnegative integer not present among* a_i*. Then, for any* $\mathcal{P}$ *position* $P_n = (a_n, b_n)$*,* $a_n = \operatorname{mex}(a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1})$ *and* $b_n = a_n + n$*. This gives a recursive formula to find* P_n*.*

This formula makes expanding the table above much simpler.

Example 8. We can easily calculate P_{10} and P_{11} using the formula. The mex of all the values in the table is 16, since we can find every integer from 0 to 15 in the table, so $a_{10} = 16$. This means that $b_{10} = a_{10} + 10 = 16 + 10 = 26$, so $P_{10} = (16, 26)$, which makes the new mex for values in the table up to $n = 10$ equal to 17. So we know that $a_{11} = 17$ and can quickly calculate that $b_{11} = a_{11} + 11 = 17 + 11 = 28$, giving us $P_{11} = (17, 28)$.

While this formula requires a table of existing $\mathcal{P}$ positions, it is a very fast way to calculate new ones. We now know how to play individual WYTHOFF'S GAME positions of any size, but we still don't know how to play in a sum of positions, or in positions with more than two piles.

4. Sprague-Grundy Values and Sums of Wythoff's games

To figure out how to play in sums of WYTHOFF'S GAME positions, we need to first know two things: how to win at NIM, and the Sprague-Grundy theorem. To learn how to win the game of NIM, we must first define an operation $\oplus$ known as the *nim sum*:

Definition 9. For numbers $a_1, a_2, \ldots, a_n$, the operation $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ equals a number b which is determined by the binary representations of $a_1, a_2, \ldots, a_n$. Each digit in the binary representation of b is a 1 if the total of 1s in the same position among a_i is odd, while the digit is 0 if the total is even.

Taking the *nim sum* is the same as adding numbers in binary without carrying over to the next column. Note that $x \oplus x = 0$ for values of x we can peform a *nim sum* on, since adding two 1s or 0s always gives a 0 in the *nim sum* system.

Example 10. Say we wanted to compute $2 \oplus 3 \oplus 5 \oplus 7$, starting by expressing all the numbers in binary form. We have $2 = 10$, $3 = 11$, $5 = 101$, and $7 = 111$. Using those representations, we can figure out the digits in the binary representation of our answer. There are 3 1s in the 2^0 position, 3 1s in the 2^1 position, and 2 1s in the 2^2 position, so our answer is $011 = 3$.

Now that we are familiar with *nim sums*, we can introduce the following, known as Bouton's theorem:

Theorem 11. *For a* NIM *position with piles of size $n_1, n_2, \dots, n_k$, the position is in $\mathcal{P}$ if $n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$, and in $\mathcal{N}$ otherwise* [**Bou01**].

This theorem gives us the winning strategy for NIM, which is: get the *nim sum* of the position down to 0 and keep it there. So for our previous example of $(2, 3, 5, 7)$, we know the piles *nim sum* to 3, so removing the pile of size 3 would leave the position at 0 and is a winning move. Now we that we know the winning strategy for NIM, we can introduce the Sprague-Grundy theorem and see why it is helpful for analyzing sums of other impartial games :

Theorem 12. ([**Spr35**], [**Gru39**]) *Given any impartial game G, the game is equivalent to a* NIM *pile of size n, which we write as $*n$. We call n the Grundy value $\mathcal{G}(G)$ of the game.*

This theorem is very useful for analyzing sums of impartial games, since we can turn them into equivalent NIM positions which are easier to analyze. The only catch is that we need a quick way to actually determine $\mathcal{G}(G)$ for any game G, which we can use the following rule for:

Theorem 13. *Given an impartial game with options $\{*n_1, *n_2, \dots, *n_k\}$, that game is equal to $*m$, where $m = \operatorname{mex}(n_1, n_2, \dots, n_k)$.*

Proof. We aim to show that $\{*n_1, *n_2, \dots, *n_k\} = *m$, which we can do by adding the two games and showing that it is a $\mathcal{P}$ position, since if they are equal then we should have $m \oplus m = 0$. The first player has two options in the game $\{*n_1, *n_2, \dots, *n_k\} + *m$, moving in $*m$ to some $*m'$ with $m > m'$, or moving to one of the n_i. In the first case, the second player can counter by moving to an $*n_i$ such that $n_i = m'$, which leaves the position at 0, winning them the game. Note that this will always be possible because m is the first number not equal to any of the n_i and $m > m'$. In the second case, the second player can move $*m$ to $*n_i$ if $m > n_i$ to get a $\mathcal{P}$ position and win. Otherwise, if $n_i > m$, then the second player can move in $*n_i$ to $*m$, which still leaves a $\mathcal{P}$ position. Note that we also have $n_i \neq m$ for all n_i by our definition of m. Thus the game $\{*n_1, *n_2, \ldots, *n_k\} + *m$ is a $\mathcal{P}$ position, which means $\{*n_1, *n_2, \ldots, *n_k\} = *m$. ■

This gives us a recursive method for calculating Grundy values, starting with $\mathcal{G}(0,0) = 0$ since $\text{mex}() = 0$. We can use our "queen on a board" game to visually model the Grundy values of WYTHOFF'S GAME, since every square the queen can occupy represents a unique position. Using the mex rule we can fill out the first few Grundy values, winding up with the following [**AN09**]:

10	11	9	8	13	12	0	15	16	17	14
9	10	11	12	8	7	13	14	15	16	17
8	6	7	10	1	2	5	3	4	15	16
7	8	6	9	0	1	4	5	3	14	15
6	7	8	1	9	10	3	4	5	13	0
5	3	4	0	6	8	10	1	2	7	12
4	5	3	2	7	6	9	0	1	8	13
3	4	5	6	2	0	1	9	10	12	8
2	0	1	5	3	4	8	6	7	11	9
1	2	0	4	5	3	7	8	6	10	11
0	1	2	3	4	5	6	7	8	9	10

If we imagine the table above to be a board, then each square contains the Grundy value of a queen placed on that square. We can see that the mex rule can be used to check the values in the board and expand the board. Our options from any given square are the squares to the left, the square below, and the squares diagonally left and below. That means the Grundy value of a square is given by the mex of the values for the squares corresponding to its options.

Example 14. We can begin to construct the Grundy values for more squares by following the mex rule. Say we wanted to figure out $\mathcal{G}(1,11)$, which would be right above the 11 in the top row. Note that $\mathcal{G} = (0,11) = 11$ since its options cover all Grundy values from from 0 to 10. With that in mind, we can see that the Grundy values covered in the options of $(1,11)$ are everything from 0 to 8, as well as 10 and 11, so $\mathcal{G}(1,11) = \text{mex}(0,1,2,\ldots,8,10,11) = 9$.

This table allows us to figure out a winning strategy for sums of WYTHOFF'S NIM positions, such as $((3,5),(2,4))$. We separate $(3,5)$ and $(2,4)$ to indicate that we are not allowed to take stones from say, the pile of 5 and the pile of 2 on the same turn. The strategy is also quite simple: use Grundy values to follow the winning NIM strategy.

Example 15. We use the table to find a winning move in $((3,5),(2,4))$. First we convert the two smaller games into NIM piles by looking up their Grundy values, finding that $\mathcal{G}(3,5) = 0$ and $\mathcal{G}(2,4) = 3$. So the equivalent NIM game is equal to $0 \oplus 3 = 3$, meaning the first player has a winning strategy. Since we want the position to be 0 after our move, the winning move is to move $(2,4)$ into a position with Grundy value 0, which we can see on the table is $(1,2)$. The position is now two games which are $\mathcal{P}$ positions in WYTHOFF'S NIM, so the first player can just follow those winning strategies in each part, taking care to always move in the same component as the opponent. Thus, $((3,5),(2,4))$ is an $\mathcal{N}$ position, and removing 3 stones from the pile of 4 is a winning first move.

Now we can begin to consider how to win when the rules themselves have been slightly altered, such as allowing 3 piles at most rather than 2.

5. Wythoff's game With Three Piles

When analyzing WYTHOFF'S GAME positions with more than 2 piles allowed, there are two different rulesets we can employ: giving the ability to take an equal number of stones from all piles, or not giving that ability. The first option may seem a little more natural, but also quickly ramps up the complexity as we start getting more piles. The second option is essentially a different type of sum than the one shown in the previous section. We will be focusing on the three-pile game where taking from all three piles in a turn is allowed if the same amount is taken from all three.

Going back to our analogy of a queen on a flat board, we can think of our three-pile game as a queen inside a cube. Whereas in 2 dimensions the queen had only 3 directions to move in, the queen in a cube has 7 directions she can move in, making the analysis of three piles much more complex. Still, we can figure out rules governing the $\mathcal{P}$ positions in the three-pile game. To start, all of the $\mathcal{P}$ positions from the two-pile game are still $\mathcal{P}$ positions of the form $(0,a,b)$. We have the following rules that three-pile $\mathcal{P}$ positions (a,b,c) with $a \leq b \leq c$ must satisfy:

- There must be no previous $\mathcal{P}$ positions of the form (a,b,c'), (a,b',c), or (a',b,c).
- If a previous $\mathcal{P}$ position is of the form (a,b',c') where $b' \neq b,c$ and $c' \neq b,c$, then it must also be true that $c'-b' \neq c-b$. Two equivalent statements can be made for previous $\mathcal{P}$ positions of the form (a',b,c') and (a',b',c), and all three statements must be true.
- For all $\mathcal{P}$ positions (a',b',c') before (a,b,c), there must not be a number n such that $(a'+n,b'+n,c'+n) = (a,b,c)$.

These rules follow directly from the available moves in three-pile WYTHOFF'S NIM and our knowledge of $\mathcal{P}$ and $\mathcal{N}$ positions. The first rule must be true because otherwise we would be able to remove some stones from a pile and end up with another $\mathcal{P}$ position, which can't be possible because all moves should go to $\mathcal{N}$ positions. The second and third rules cover moves which remove stones from 2 piles and 3 piles, respectively. Using these rules we can begin to find $\mathcal{P}$ positions with three piles of stones, of which the first four are $(1, 1, 4), (1, 3, 3), (2, 2, 6)$ and $(3, 4, 4)$.

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