APPLICATIONS OF COMBINATORIAL GAME THEORY TO POPULAR COMBINATORIAL GAMES

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1. Abstract

In this paper, I will talk about the applications that theorems and definitions of combinatorial game theory have to the popular combinatorial game of chess. Although, combinatorially speaking, chess is a fairly complicated game, this paper will break it down into easier scenarios, where combinatorial game theory can be applied. When broken down like this, there are various aspects of combinatorial game theory which will be usefully applied to chess.

2. Introduction

There are multiple concepts which need to be understood when talking about combinatorial game theory with respect to these popular games. First of all, numbers are often used to characterize games in relation to combinatorial game theory. A positive number alludes to a winning outcome for left (first to move), while a negative number alludes to a winning outcome for right (second to move).

Second of all, option notation is a way of describing a game with two parts; the set of options left has, and the set of options right has. From option notation, it can quickly be seen which side has an advantage. Option notation is denoted as $\{ \mid \}$, with numbers potentially on either side of the middle line. In this paper, both specific numbers and different option notations will repeatedly be mentioned in order to convey both players current situations.

This paper assumes the reader is familiar with the game of chess. As is obvious, a regular game of chess can end in a stalemate, or draw. However, to qualify as a combinatorial game, there must be no draws. Therefore, we must take a liberty in order to talk about chess as a combinatorial game. In this paper, we will operate under the assumption that the player who runs out of moves first loses the game. This will not have an affect on applying the methods we talk about to real games.

3. Chess

3.1. **Combinatorial Chess.** Chess can be an extremely confusing game to discuss in a combinatorial manner. This is primarily due to the long moving pieces, which can move to all areas of the board. Simple calculations show that in just the first four moves of a chess game, there are approximately 318,979,564,000 possible moves in total. In addition to this, experts have been unable to discover which side has an advantage at the beginning of a game, or even whether a stalemate is the outcome if both sides play optimally throughout the game. This is part of the beauty of chess; both players can win the game, and so much depends on the openings. However, from a combinatorial viewpoint, this creates a nightmare. We are unable to assign a starting value to a game of chess, resulting in no valid option notations either. Even after an opening is played by both sides, the wide range of possibilities resulting from bishops and queens on each side make it impossible to efficiently discuss a chess game in a combinatorial manner. It is not until a chess game reaches endgame that combinatorial game theory can truly be applied.

3.2. **Endgame.** Although there are general types of endgames in which we can talk about, a variety of them are rather straightforward. In the case which one color is left solely with a king, they obviously have no winning scenarios left. Does this mean the other player no longer has optimal moves? Basic chess knowledge tells us that the best way to proceed when left with solely a King is to try and force a stalemate, a position where you no longer have any moves left, and by virtue, the game ends as a draw. Therefore, there are still optimal moves left for both sides, meaning we can write option notation. Although this paper is too

short to include chess boards to portray specific scenarios, we can still talk about different situations with a decent amount of specificity.

3.3. **The Zero Game.** Earlier, the concept of defining combinatorial games as values was introduced. In combinatorial game theory, we define the zero game as a game where neither player can make a move that abides by the rules. We write this as Game $= 0 \{ \mid \}$. In regular chess, there is no possible scenario attained by normal game play where both players are left with no rule-abiding moves. However, let's imagine a chess board such as figure 3.1.

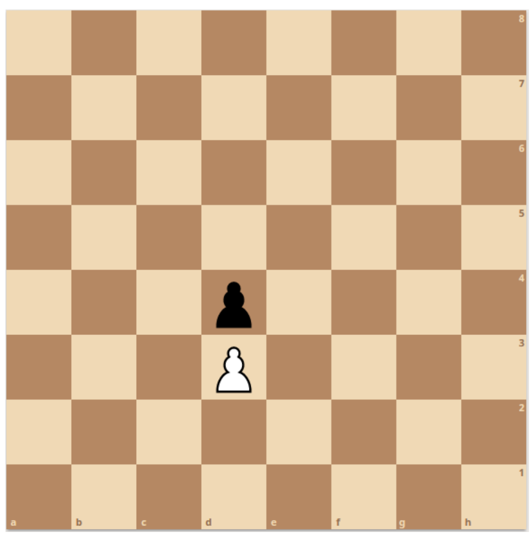

Figure 3.1 - a zero game

Although the absence of kings means we will obviously never see a situation such as this in real chess, this is a zero game created with pawns- neither player is able to make a lawful move. Another attribute of this zero game is that whichever player's move it is automatically loses (operating under the assumption that the player left with no moves loses). We can now make chess boards which are equivalent to this 0 position.

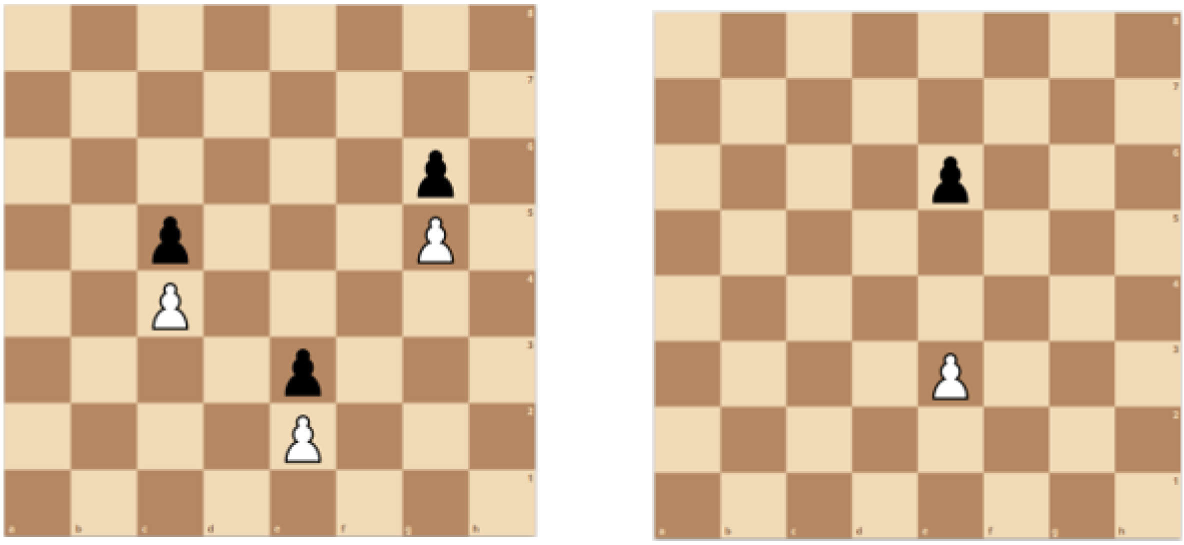

Figure 3.2 - more examples of a zero game

By analyzing each above scenario, we can see that the first player to move will run out of moves first. Therefore, each of these above chess scenarios can be assigned a value of 0.

3.4. **Assigning positive and negative values.** The most basic combinatorial game other than 0 is 1. 1 is written in option notation as $1 = \{0 \mid \}$. This signifies that left can move to a winning position while right has no moves. What would this look like in a game of chess? White would be able to move to a 0 position as discussed above, while black's only move would result in a white win. Since the board obviously isn't symmetrical at this point, we know this white win must come from virtue of a pawn capture and queen promotion.

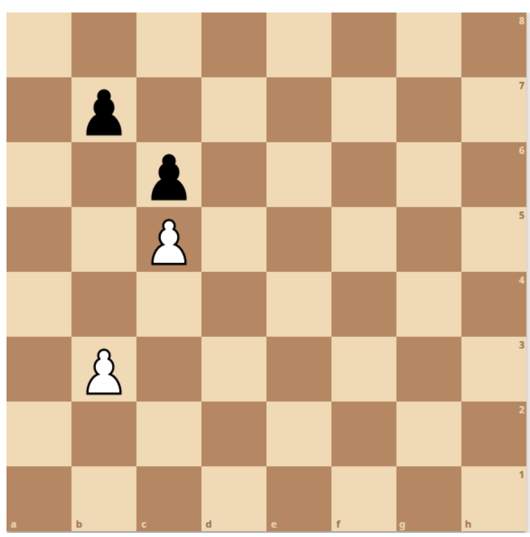

Figure 3.3 - example of a 1 game

As can be seen from this 1 game, white is able to move to a 0 position, while black's only move leads to a white victory.

3.5. **Including Kings.** Although playing around with scenarios including solely pawns can be fun, the inclusion Of Kings is the only way we will actually be applying combinatorial game theory to chess. A 0 game which includes Kings is one that either player to move loses.

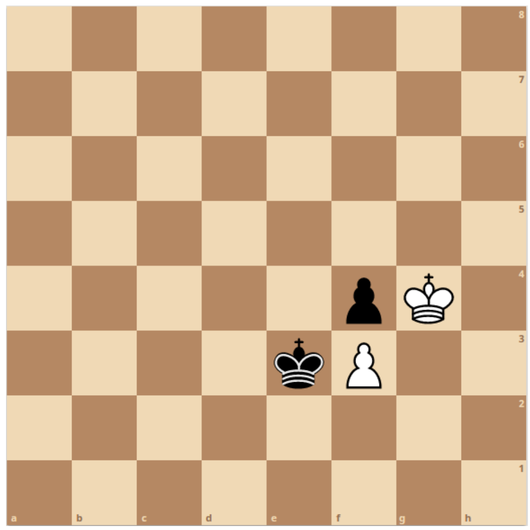

Figure 3.4 - zero game with King

As can be seen in the above scenario, either side to move will automatically lose the game, as their pawn will be taken, and the other side can promote and checkmate with optimal play.