In the game of Angels and Devils

STRATEGIES TO WIN AS THE ANGEL

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1. Introduction to the Angel problem

The Angel problem, commonly referred to as the Angels and Devils game, is a question proposed by John Conway [4]; The game is played on an infinite chess board with two players, known as the Angel and the Devil.

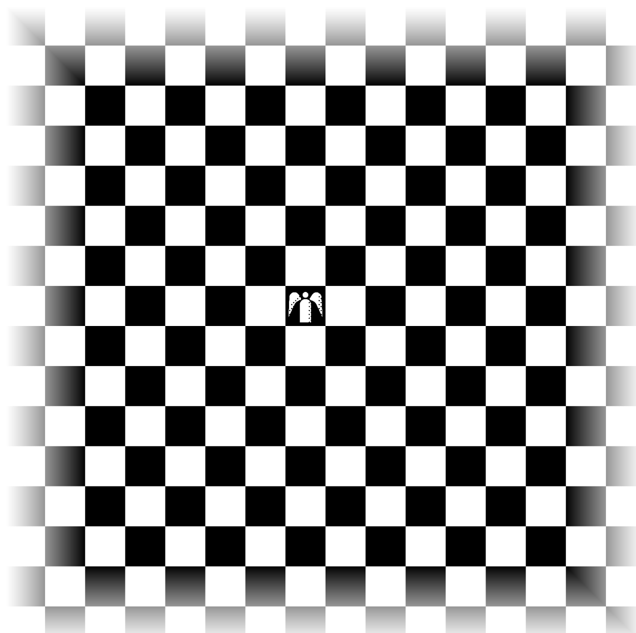

Figure 1. The Angel on an infinite chessboard.

The Angel has the power to move k times, which is stated before the game begins. The board starts empty with the Angel occupying on square. On every turn, the Angel moves to a different square that could be reached within k moves of a chess king. Then the Devil may add a block on any square not currently occupied by the Angel.

(1) The Angel cannot land on a blocked square.
(2) The Angel can make their move regardless of if there is a block in the way, meaning the Angel can "leap" over blocked squares.

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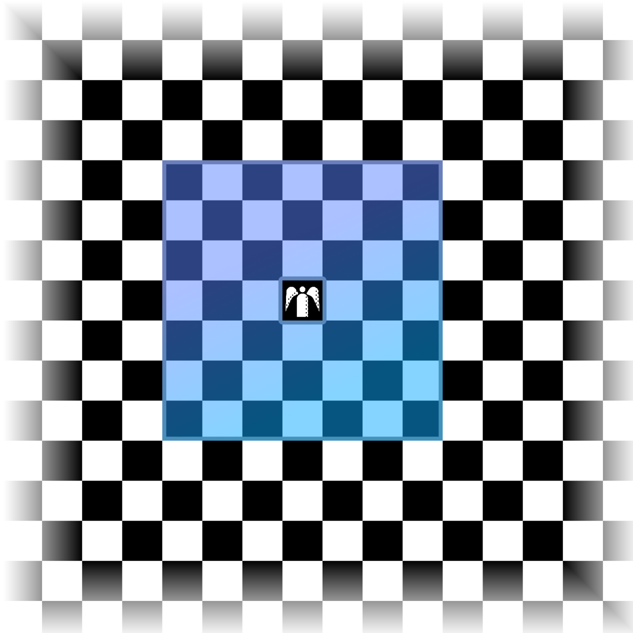

Figure 2. The blue region represents the squares an Angel with a power of 3 can reach with one move.

The Devil wins if the Angel is trapped and unable to move. Otherwise, the Angel wins by surviving indefinitely. The problem itself, proposed by Conway, is to find the lowest possible number k can be for the Angel to win.

2. Intuitive escape strategies

There are many seemingly intuitive escape strategies for the Angel. One such example is for the Angel to move in one direction while also avoiding any obvious traps by moving to the side. While it may seem at first glance like the Devil would never be able to win, this strategy is actually winning for the Devil.

Observe that, if the Angel moves in one direction as fast as possible, that the Angel's future positions lie in a in the shape of a cone. The Devil can simply build a wall that the Angel can not leap over far away in the path of the cone. When the Angel finally arrives, the wall is complete and because the Angel chooses to move exclusively in one direction, they have nowhere to go and therefore the strategy fails.

The Angel may also intuitively assume that moving away from nearby blocks will be winning for the Angel. However, this is not the case. The Devil can simply construct a wall in the shape of an arc far north, then nudge the Angel into it by blocking the square right to the south of the Angel. If the Angel tries to avoid the faraway trap, then the Devil can block faraway squares to the south, effectively blocking the Angel and forcing him to go into the arc.

Each of these strategies will fail no matter what the value of k is.

3. How to actually win as the Angel

Máthé [2] introduces the nice Devil who never blocks a square on which the Angel has previously stayed nor a square on which the Angel could have possibly jumped at on a previous turn, but did not. The Angel, when playing against the nice Devil, will acknowledge defeat when he is trapped in a finite region of the chessboard. Otherwise, the Angel would be able to oscillate back and forth between two positions and never lose. Let us call Máthé's version of the Angels and Devils game *variant 1*.

The Angel wins against the nice Devil by:

(1) Playing as if the entire left side of the board ($x < 0$) is blocked by the devil, on top of any squares the devil has actually taken.
(2) Skirting along the blocked squares, adopting a ”Hand on the wall” technique, keeping a left hand on the blocks, and leaving a trail. We can call an Angel that moves as such a *runner*.
(3) Leaping on a square that already has a trail left on it effectively creates a circle. The angel wins by following this trail, in a circle, indefinitely.

In order to check that an angel of sufficient power k against the nice Devil also has sufficient power k against the real Devil, we will have to make a series of statements.

Claim 3.1. *If the Devil can catch the Angel of power k, then there exists a positive integer N such that the Devil can entrap the Angel in $B(N)$.*

Proof. If for every positive integer n the Angel has a strategy to make n moves without jumping on eaten squares, then a suitable limit of these strategies gives a winning strategy for the Angel to move forever without jumping on eaten squares. Hence if the Devil can catch the Angel, then there exists an n such that the Devil can catch the Angel in at most n steps. Thus, the Devil can entrap the Angel of power k in $B(np)$. ■

Theorem 3.2. *For any positive integers k and N, if the Devil can entrap the Angel of power k in $B(N)$, then the* nice Devil *can also entrap the Angel of power k in $B(N)$.*

Before proving the theorem precisely, we need to discuss what a strategy for the Devil is.

Definition 3.3. A *journey* of the Angel of power k is a finite sequence of squares $(v_0, v_1, ..., v_n)$ such that $v_0 = (0, 0)$ and $d(v_i, v_{i+1}) \leq k$ for each $0 \leq i < n$.

Hence a Devil's strategy is some function Φ which maps every (finite) *journey* of the Angel to some square (which is to be eaten). To be more precise, Φ maps from the *journey*s to $\mathbb{Z}^2 \cup$ nothing, since we also allow the Devil not to eat anything if he wishes. (This way the Devil also knows which squares are already eaten on the board, so the *journey* of the Angel holds all the information on the game.)

We say that a *journey* of the Angel is allowed against some Devil strategy if the Angel following this *journey* has never jumped on an eaten square with respect to this strategy of the Devil.

Proof of Theorem 3.2. Fix a strategy Φ for the Devil with which he can entrap the Angel of power k in $B(N)$ (that is, the Angel cannot leave this domain without jumping on eaten squares).

Let (v0, v1, . . ., vn) be a *journey.*

We define a directed graph on $G = 0, 1, ..., n$. For each $i \in G$, $i \neq 0$, let j be the least non-negative integer for which $d(v_j, v_i) \leq k$. (Thus $j < i$, since $d(v_{i-1}, v_i) \leq p$.) We connect i to j by a directed edge for each $1 \leq i \leq n$. So the graph G has n directed edges. There is a unique path from n to 0. Let us denote this path by $(a_p, a_{p-1}, ..., a_0)$ where p is the number of vertices in this path. Thus $a_p = n$ and $a_0 = 0$. For each $0 \leq i \leq p$ let

$$u_i = v_{a_i}.$$

We call the sequence $(u_0, u_1, ..., u_p)$ the reduced *journey* of the *journey* $(v_0, v_1, ..., v_n)$, provided that $v_n \neq v_0$. If $v_n = v_0$ then the reduced *journey* is defined to be (u_0) and p is defined to be 0. Notice that $u_0 = (0,0)$, $u_p = v_n$, and $d(u_i, u_{i+1}) \leq k$ for each $0 \leq i < p$. Moreover, for each $1 \leq i \leq p$, if j is the smallest index for which $d(v_j, u_i) \leq k$, then $u_{i-1} = v_j$. Hence for each two distinct i and j we have $u_i \neq u_j$.

Now we shall define the *nice Devil*'s winning strategy Ψ. The *nice Devil* on his first turn (while the Angel is still at $(0,0)$) eats the same square the Devil would eat on his first turn (which is $\Phi(((0,0)))$ to be precise), except if the Devil would eat $(0,0)$, in this case the *nice Devil* eats nothing. Given an arbitrary *journey* $(v_0, v_1, ..., v_n)$, let us denote its reduced *journey* by $(u_0, u_1, ..., u_p)$, let $z = \Phi((u0, u1, ..., uk))$, and then define

$$\Psi((v_0, v_1, ..., v_n)) = \begin{cases} z \text{ if } d(z, v_l) > p \text{ for each } 0 \leq l < n \text{ and } z \leq (0,0), \\ \text{nothing otherwise.} \end{cases}$$

Thus the *nice Devil*'s strategy (Ψ) is to eat the square the Devil (Φ) would eat for the reduced *journey*, but only if he is allowed to eat this square, otherwise he eats nothing; since he is nice.

We claim that Ψ is a winning strategy for the *nice Devil*; that is, the Angel cannot leave the domain $B(N)$ whilst playing against Ψ.

■

From Claim 3.1 and Theorem 3.2 we obtain the following:

Theorem 3.4. *If the Devil can catch the Angel of power k, then there is an N such that the* nice Devil *can entrap the Angel of power k in the domain $B(N)$.*

And, with the following:

Proposition 3.5. *If k is sufficiently large, then the* runner *never goes back to the origin. Moreover, the* runner *travels arbitrarily far from the origin, whatever the* nice Devil *does.*

The proof is based on the fact that if k is sufficiently large then "the runner *runs faster than the* nice Devil *can build walls".*

Proof. Notice that if the Runner does not go back to the origin, then by the fact that, if at least one wall has been painted twice, then the first wall painted twice is the first wall the Runner painted (This can be proven by the fact that there cannot be any edge before the first wall painted that could serve as the wall which is not the first wall the Runner painted being painted twice), every wall is painted at most once.

Suppose that $k \geq 11$. It is easy to see that whatever the *nice Devil* does in his first move, after the first turn of the Runner he is at least k high to the north. Also, after the second turn of the Runner, he is at least $2k$ high to the north.

We shall prove by induction on t that the Runner does not go back to the origin in t steps. For $t = 2$ we are done. Suppose that the Runner didn't go back to the origin in t steps. After t steps, the *nice Devil* has eaten up at most t squares, which altogether have at most $4t$ walls. The Runner of power k paints at least tk walls during this time, and these are different walls by our assumption. Thus, the Runner has painted at least $tk - 4t$ walls on the pre-eaten squares (along the y axis). Along the pre-eaten squares he can only go north but not south. Thus, once during this time he was at least $tk - 4t$ high up to the north. He could not go down to the south more than t walls, so after t steps his north coordinate is at least

$$tk - 4t - t = t(k - 5).$$

Since $t \geq 2$ and $k \geq 11$, this is at least $2(k-5) \geq k+1$. Thus, the Runner will not go back to the origin in $t+1$ steps: his north coordinate will be at least one. Hence, by induction, we also get that for each t, after t steps the Runner is at least $t(k-5)$ high to the north. Thus the Runner travels arbitrarily far from the origin. ■

Proposition 3.5 clearly implies that the Angel of sufficiently large power can defeat the *nice Devil*. Thus from Theorem 3.4 we obtain the following:

Theorem 3.6. *The angel of sufficiently large power can defeat the devil.*

Proposition 3.7. *The Angel's trail will never reach the x axis again.*

Proof. Consider the first occasion the trail touched the x axis. Suppose that this happened in the Angel's tth turn. Hence the *nice Devil* has eaten already, at most, t squares, and the Angel has painted already at least $2(t-1)+1$ walls: $2(t-1)$ in his first $t-1$ moves, and he had to paint at least one more in this turn to reach the x axis. It is easy to conclude from the preceding that these are different walls. Let d be the number of squares the *nice Devil* ate in the region $(x, y) : x \geq 0, y \geq 0$. Let a be the number of different walls the Angel painted till his trail reached the x axis. Thus

$$d \leq t \text{ and } a \geq 2t - 1.$$

Let us interrupt the game at this point. Let us delete everything from the lower half of the board: that is, delete all the eaten and the pre-eaten squares of $(x, y) : y < 0$. There is no trail on this half plane. Let us also delete all pre-eaten squares except for those $(-1, y) : 0 \leq y \leq N - 1$ for some very large integer N. Let $N - 100$ be larger than the $y$coordinate of the uppermost eaten square which the *nice Devil* ate during the game, and also larger than the y coordinate of the uppermost point of the trail. Hence there remained exactly $N + d$ eaten squares on the board.

Now let us ask the Angel to continue his *journey* and his trail-painting as he did before. But we will not allow the *nice Devil* to eat any more squares. Sooner or later the Angel will return to his starting position in the square $(0, 0)$, and the green line will return to the point $(0, 0)$. Hence the green line will form a circle. Let us denote its length by l, which is also the total number of walls painted. ■

Theorem 3.8. *The Angel of power 2 can defeat the* nice Devil*.*

Proof. We have given a strategy for the Angel of power 2 such that the trail never reaches the x axis again, and thus the Angel never paints the same wall again (Proposition 3.7). Since in a bounded domain there are only routes of finite length where the trail could go, obviously the trail will leave every bounded domain—hence the Angel also will leave every bounded domain, however the *nice Devil* plays. ■

Thus by Theorem 3.4 we immediately obtain the following.

Theorem 3.9. *The Angel of power 2 can defeat the Devil.*

References

[1] John. H. Conway. *The Angel problem* Games of No Chance, 29:3-12, 1996.

[2] András Máthé. *The Angel of power 2 wins.* Combinatorics, Probability and Computing, 16(3):363-374, 2007.

[3] Brian H. Bowditch. *The Angel game in the plane* Combinatorics, Probability and Computing, 16(3):345-362, 2007.

[4] Berlekamp, Elwin R.; Conway, John H.; Guy, Richard K. *The King and the Consumer* Winning Ways for your Mathematical Plays, Volume 2: Games in Particular, Academic Press, 2(19):607-634, 1982.