Classical Impartial Games

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August 15, 2021

1 Introduction

An impartial combinatorial game is a 2-player combinatorial game in which the same moves are available to each player. For example, the game Nim consists of many piles of tokens, and any number of tokens can be removed from a particular pile by either player. As we have learned in [Rub], to determine whether the next player to move will win in a Nim game, we shall compute the XOR sum of the number of tokens in all piles in the game. If the result is greater than 0, the next player to move can win; the winning strategy is to make a move that changes the XOR sum of all the piles to 0. On the other hand, if the result is equal to 0, then the next player to move will lose, assuming both players play optimally.

In this paper, we will analyze in a similar manner two other impartial combinatorial games: Fibonacci Nim and Wythoff's Game.

2 Impartial Games

Since we are studying impartial games, it would be useful to first review some basics about impartial combinatorial games:

- An $\mathcal{N}$ position is a game that is winning for the next player.
- A $\mathcal{P}$ position is a game that is winning for the previous player (or losing for the next player).
- The only outcomes of an impartial combinatorial game are $\mathcal{N}$ and $\mathcal{P}$.

We also have our favorite Partition Theorem, which will be very useful in this paper (see pg. 50 of [Rub]). In addition, we can see that the Sprague-Grundy Theory can be used to analyze our games. Before we review the Sprague-Grundy Theory, we will need to define the "minimal excludant" (mex) operation:

Definition 2.1. Let S be a finite set of non-negative integers. Then, $mex(S)$ is the least non-negative integer that is not in S.

Here are some of the basic rules of the Sprague-Grundy Theory:

1. We can assign a non-negative Grundy value to each game position.
2. If the Grundy value of a game is equal to 0, that game is a $\mathcal{P}$-position. Else, the game is an $\mathcal{N}$-position.

3. The Grundy value of a game can be found recursively by performing the *mex* operation on the Grundy values of its options.

4. To derive the Grundy value of a disjoint sum of games, we take the Nim sum ($\oplus$) of the components.

3 Fibonacci Nim

Definition 3.1. Fibonacci nim is an impartial combinatorial game played on a single heap of, say, n stones. To start, the first player can remove any number of stones between 1 and $n-1$. After the first move, any player can move as long as the following condition is satisfied: if the previous player removed r stones, then the next player can remove at most $2r$ stones. The last player to make a move wins.

Example 3.1.1 (Demonstration of Fibonacci Nim)**.** We start with a heap of 5 stones. Left takes 1 stone, leaving 4 stones. Since Left took 1 stone, Right can take at most 2 stones on his turn. Right takes 1 stone, leaving 3 stones. Since Right took 1 stone, Left can take at most 2 stones. Left takes 2 stones, leaving 1 stone. Right takes the last stone and wins the game.

This game is not as simple as the game Nim. In Nim, the players can remove any number of stones from a pile. However, in Fibonacci Nim, the number of stones that each player can remove on their turn is limited by how many stones the previous player removed. So, a game position depends not only on how many stones are in the heap but also on how many stones the next player can remove. Hence, we can encapsulate a game position as follows:

Definition 3.2. Let the ordered pair (n, r) represent a game position, where n is the number of stones in the heap, and r is the maximum number of stones the next player can remove from that heap.

Given a game (n, r), we can list out the possible moves that the next player can make: the next player could take either 1 stone, 2, stones, ..., or r stones. So, we can represent the game (n, r) in terms of option notation, where the options are also ordered pairs as defined in Definition 3.2:

$$(n, r) = \{(n-k, 2k) : 0 < k \leq r\} \tag{1}$$

Each option of (n, r) is of the form $(n-k, 2k)$ because, if the next player removes k stones from the heap, the total number of stones left would be $n-k$, and the player after him/her can remove up to $2k$ stones. Every game with n stones starts out as $(n, n-1)$.

We can apply the Sprague-Grundy theory to analyze Fibonacci Nim. Since we know the options of each game state, we can recursively find the Grundy value of a game state by computing the *mex* of the Grundy values of its options:

$$\mathcal{G}(n, r) = mex(\mathcal{G}(n-k, 2k) : 0 < k \leq r) \tag{2}$$

Using this recursive method, we can build a table of Grundy values, as shown below in Figure 1. Note that the Grundy values are omitted for the game positions where

$n < r$ because these games are pretty simple: the first player is able to remove all of the stones in the pile and win in the first turn itself.

$n\backslash r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0																				
1	0	1																			
2	0	0	2																		
3	0	0	0	3																	
4	0	1	1	3	3																
5	0	0	0	0	0	4															
6	0	1	1	1	1	4	4														
7	0	0	2	2	2	4	4	4													
8	0	0	0	0	0	0	0	0	5												
9	0	1	1	1	1	1	1	1	5	5											
10	0	0	2	2	2	2	2	2	5	5	5										
11	0	0	0	3	3	3	3	5	5	5	5	5									
12	0	1	1	3	3	3	3	3	6	6	6	6	6								
13	0	0	0	0	0	0	0	0	0	0	0	0	0	6							
14	0	1	1	1	1	1	1	1	1	1	1	1	1	6	6						
15	0	0	2	2	2	2	2	2	2	2	2	2	2	6	6	6					
16	0	0	0	3	3	3	3	3	3	3	3	3	7	7	7	7	7				
17	0	1	1	3	3	3	3	3	3	3	3	3	3	7	7	7	7	7			
18	0	0	0	0	0	4	4	4	4	4	4	7	7	7	7	7	7	7	7		
19	0	1	1	1	1	4	4	4	4	4	4	4	7	7	7	7	7	7	7	7	
20	0	0	2	2	2	4	4	4	4	4	4	4	4	7	7	7	7	7	7	7	7

Figure 1: Table of Grundy values for FIBONACCI NIM, taken from [LS15]

In this paper, we will only focus on finding the $\mathcal{N}$ and $\mathcal{P}$ positions, or the positions whose Grundy value is greater than 0 and the positions whose Grundy value is equal to 0, respectively.

In order to find the $\mathcal{N}$ and $\mathcal{P}$ positions in FIBONACCI NIM, we have Zeckendorf's theorem:

Theorem 3.1 (Zeckendorf's theorem)**.** *Every positive integer has a unique representation as a sum of distinct Fibonacci numbers such that no two numbers are consecutive in the Fibonacci number sequence. This representation of a number is called its Zeckendorf representation.*

See [21] for the proof of Zeckendorf's theorem.

Next, we will introduce new notation. First, we write F_n to denote the n^{th} Fibonacci number. Also, we define $z_i(n)$ be the i^{th} smallest element in the Zeckendorf representation of n. For example, $z_1(4)$ is 1, and $z_2(26) = 21$. We will soon learn in Theorem 3.2 that, to win in a FIBONACCI NIM game (n, r), we must remove $z_1(n)$ from the pile of n stones (that is, if $z_1(n) \leq r$).

Theorem 3.2. $\mathcal{G}(n, r) = 0$ *if and only if* $r < z_1(n)$.

First, we need to prove a lemma that will help us prove Theorem 3.2:

Lemma 3.3. *Let n be an integer greater than 1, and k be an integer such that $1 \leq k < z_1(n)$. If $z_1(k) = F_t$, then $z_1(n-k)$ is either F_{t+1} or F_{t-1}.*

Proof from [LS15]. We prove this through induction on the number of elements in the Zeckendorf representation of k.

Let's start with the base case, or when k is a Fibonacci number itself. Let k be F_t. Then, suppose that $z_1(n) = F_s$. Then, there are two cases related to the parity of s and t: $s \equiv t \pmod 2$ and $s \not\equiv t \pmod 2$.

If $s \equiv t \pmod 2$, then we can write t as $s - 2d$ for some $d \geq 1$, and we have

$$\begin{aligned} F_s - k &= F_s - F_{s-2d} \\ &= (F_{s-1} + F_{s-3} + F_{s-5} + ... + F_{s-2d+3} + F_{s-2d+1} + F_{s-2d}) - F_{s-2d} \\ &= F_{s-1} + F_{s-3} + F_{s-5} + ... + F_{s-2d+3} + F_{s-2d+1} \end{aligned}$$

Here, we have found the Zeckendorf representation of $F_s - k$. To find the Zeckendorf representation of $n - k$, we can rephrase $n - k$ as $(n - F_s) + (F_s - k)$. We know the representation of $F_s - k$, and that the greatest element in this representation has to be less than F_s. Next, we can derive the representation of $n - F_s$ by removing F_s from the representation of n, and we see that the least element in the representation of $n - F_s$ must be greater than F_s (since $z_1(n) = F_s$). So, because the representations of $n - F_s$ and $F_s - k$ don't overlap, we can "concatenate" the two representations together to get the representation of $n - k$. Finally, we can say that, in the Zeckendorf representation of $n - k$,

$$\begin{aligned} z_1(n - k) &= z_1(F_s - k) \\ &= F_{s-2d+1} \\ &= F_{t+1} \end{aligned}$$

Now let's explore the other case, where $s \not\equiv t \pmod 2$, then we can write t as $s - 2d - 1$ for some $d \geq 0$, and we have

$$\begin{aligned} F_s - k &= F_s - F_{s-2d-1} \\ &= (F_{s-1} + F_{s-3} + F_{s-5} + ... + F_{s-2d+1} + F_{s-2d-1} + F_{s-2d-2}) - F_{s-2d-1} \\ &= F_{s-1} + F_{s-3} + F_{s-5} + ... + F_{s-2d+1} + F_{s-2d-2} \end{aligned}$$

Again, we have found the Zeckendorf representation of $F_s - k$, and we can observe that $z_1(F_s - k) = F_{s-2d-2}$. As we found in the previous case, we can see that $z_1(n-k)$ must be equal to $z_1(F_s - k)$, so we get that

$$\begin{aligned} z_1(n - k) &= z_1(F_s - k) \\ &= F_{s-2d-2} \\ &= F_{t-1} \end{aligned}$$

Now, assume that the induction hypothesis holds when the Zeckendorf representation of k has $p - 1$ elements. Then we have to prove that, when the representation of k has p elements, $z_1(n - k) = F_{t+1}$ or F_{t-1}.

Let $z_1(k)$ be F_t. We get that $z_1(k - z_1(k)) = z_2(k) \geq F_{t+2}$. Since $k - z_1(k)$ has $p - 1$ parts, we know that $z_1(n - (k - z_1(k))) \geq F_{t+1}$ from the induction hypothesis, and consequently we get that

$$\begin{aligned} & z_1(n - (k - z_1(k))) \geq F_{t+1} > F_t = z_1(k) \\ \implies & z_1(n - k + z_1(k)) > z_1(k) \end{aligned}$$

This inequality is really of the form $z_1(n') > k'$, where n' is $n - k + z_1(k)$ and k' is $z_1(k)$. Since k' has only one component in its own Zeckendorf representation, we can use the base case to show that $z_1(n' - k') = F_{t+1}$ or F_{t-1}. We can simplify $n' - k'$ to $n - k$, and we get that $z_1(n - k) = F_{t+1}$ or F_{t-1}. ■

Now, let us prove Theorem 3.2:

Proof from [LS15]. Let $\mathscr{P}$ be the set of games (n, r) such that $z_1(n) > r$, and let $\mathscr{N}$ be the set of games (n, r) such that $z_1(n) \leq r$. We need to show that:

1. In every game in $\mathscr{P}$, every move leads to a game in $\mathscr{N}$.
2. In every game in $\mathscr{N}$, there is at least one move to a game in $\mathscr{P}$.

If these two statements are true, then $\mathscr{N}$ is the set of all $\mathcal{N}$ positions, and $\mathscr{P}$ is the set of all $\mathcal{P}$ positions, according the the Partition Theorem.

In the first part of the proof, we need to show that, in a game (n, r) with $z_1(n) \leq r$, there must be a move to a game $(n-k, 2k)$, by removing k stones, such that $z_1(n-k) > 2k$. We show that $z_1(n - z_1(n)) > 2z_1(n)$, where k is substituted with $z_1(n)$. Let $z_1(n)$ be F_t. Since the Zeckendorf representation of a number must not contain consecutive Fibonacci numbers, the value of $z_2(n)$ must be at least F_{t+2}, so we get that

$$z_2(n) \geq F_{t+2} = F_{t+1} + F_t > 2F_t = 2z_1(k)$$
$$z_2(n) > 2z_1(k)$$

We also know that $z_2(n) = z_1(n - z_1(n))$. Substituting, we get

$$z_1(n - z_1(n)) > 2z_1(k)$$

Hence, when $k = z_1(n)$, removing k stones from game (n, r) is a move to a game in $\mathscr{P}$.

In the second part of the proof, we must show that, in a game (n, r) with $z_1(n) > r$, all possible moves must lead to a game $(n - k, 2k)$ such that $z_1(n - k) \leq 2k$. Given that $z_1(n) > r$ and that all possible values of k must be less than or equal to r, we get that $k < z_1(n)$. Thus, by Lemma 3.3, if we let $z_1(k)$ be F_t, then $z_1(n - k) \leq F_{t+1}$. Using this information, along with the fact that $F_t = z_1(k) \leq k$, we get that

$$z_1(n - k) \leq F_{t+1} = F_{t-1} + F_t \leq 2F_t \leq 2k$$
$$\implies z_1(n - k) \leq 2k$$

Hence, for every value of k, or, for every move that the next player can make, that move will be to a game in $\mathscr{N}$.

Finally, we reach the conclusion that $\mathscr{N}$ must be the set of all $\mathcal{N}$-positions, and $\mathscr{P}$ must be the set of all $\mathcal{P}$-positions, due to the Partition Theorem. ■

Here, we have shown that, given a game (n, r), the winning strategy for the next player is to remove $z_1(n)$ stones from the heap of n stones, and this is possible only when $z_1(n) \leq r$. Otherwise, the next player to move will lose, assuming that both players play optimally.

Corollary 3.3.1 (from [LS15])**.** *In the game* $(n, n-1)$*, which is the starting position of a game with n stones,* $\mathcal{G}(n, n-1) = 0$ *if and only if n is a Fibonacci number.*

For the case that Fibonacci Nim is played on multiple heaps of stones, we could apply the Sprague-Grundy theory in order to analyze such games, but that can only be done under the condition that each heap is completely independent of each other. In Fibonacci Nim, the possible moves that the next player can make not only depends on the physical state of the game but also on the moves that have been made before. This gives us two ways to restrict the options of the players:

1. Global move dynamic: If the previous player removed r stones from a heap, then the next player can remove only at most $2r$ stones from any heap.
2. Local move dynamic: As the game progresses, store the move history for each heap. For a given heap, if the player who played last in it (which could be either Left or Right) removed r stones from that heap, then the next player can remove at most $2r$ stones from that particular heap. Note that the number of stones that can be removed from a heap can vary across all heaps.

Using the local move dynamic allows us to treat each heap as disjoint components, as opposed to the global move dynamic, so we can apply the Sprague-Grundy theory with the local move dynamic. We can evaluate the Grundy value of each component, and, to find the Grundy value of a disjoint sum of components, we can take the Nim sum ($\oplus$) of the Grundy values of the components.

4 Wythoff's Game

Definition 4.1. Wythoff's game is an impartial combinatorial game played on two heaps of stones. Each player, on his/her turn, has one of the following choices: removing stones from the first heap, removing stones from the second heap, or removing the same number of stones from both heaps. The last player to make a move wins.

Example 4.1.1 (Demonstration of Wythoff's Game)**.** We start with two heaps, one with 5 stones and one with 8 stones. Left removes 1 stone from each pile, leaving 4 and 7 stones. Right removes 3 from the first pile, leaving 1 and 7 stones. Left removes 5 stones from the second pile, leaving 1 and 2 stones. Right removes 1 stone from the second pile, leaving 1 and 1 stones. Left takes both stones and wins the game.

This game has an analogous representation with a queen on a chessboard. The queen's row number represents the number of stones in the first heap, and the queen's column number represents the number of stones in the second heap. The players can either move the queen left, down, or diagonally towards the bottom-left corner of the board. The player who is unable to make a move loses, or, in other words, the player who makes the move to the bottom-left corner of the board wins. A picture of this game is shown below in Figure 2:

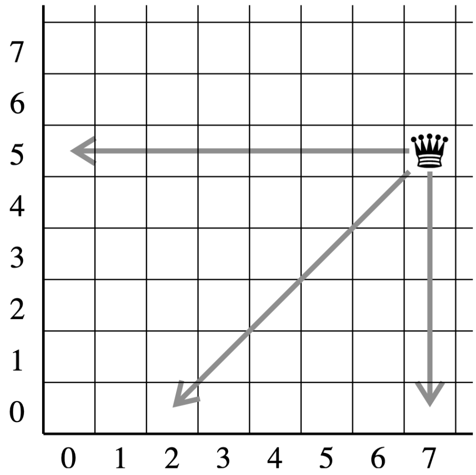

Figure 2: Wythoff's Game on a chessboard, taken from [Niv04]

We can encapsulate a game state as follows:

Definition 4.2. Let the ordered pair (x, y) represent a game position, where x the number of stones in the first heap, and y is the number of stones in the second heap.

Notice that the game (x, y) can also be thought of as the position of the queen on the chessboard. In this paper, when we describe an arbitrary game (x, y), we will frequently refer to x as the column number and y as the row number.

Given a game (m, n), the next player to move can either take a number of stones from the first pile, a number of stones from the second pile, or the same number of stones from both piles. So, we can definitely create an option set for this game:

$$\begin{aligned}(m, n) = &\{(m-k, n) \mid 1 \leq k \leq m\} \\ &\cup \{(m, n-k) \mid 1 \leq k \leq n\} \\ &\cup \{(m-k, n-k) \mid 1 \leq k \leq min(m, n)\}\end{aligned} \tag{3}$$

Since Wythoff's Game is an impartial combinatorial game, we can apply the Sprague-Grundy theory for analysis. Since we know the options of each game state, we can recursively find the Grundy value of a game state by computing the *mex* of the Grundy values of its options:

$$\begin{aligned}\mathcal{G}(m, n) = mex(&\{\mathcal{G}(m-k, n) \mid 1 \leq k \leq m\} \\ &\cup \{\mathcal{G}(m, n-k) \mid 1 \leq k \leq n\} \\ &\cup \{\mathcal{G}(m-k, n-k) \mid 1 \leq k \leq min(m, n)\})\end{aligned} \tag{4}$$

Using this recursive method, we can build a table of Grundy values, as shown below in Figure 3. Notice that the table is symmetric about the $y = x$ axis, because a game (m, n) is virtually identical to the game (n, m).

15	15	16	17	18	10	13	12	19	14	0	3	21	22	8	23	20
14	14	12	13	16	15	17	18	10	9	1	2	20	21	7	11	23
13	13	14	12	11	16	15	17	2	0	5	6	19	20	9	7	8
12	12	13	14	15	11	9	16	17	18	19	7	8	10	20	21	22
11	11	9	10	7	12	14	2	13	17	6	18	15	8	19	20	21
10	10	11	9	8	13	12	0	15	16	17	14	18	7	6	2	3
9	9	10	11	12	8	7	13	14	15	16	17	6	19	5	1	0
8	8	6	7	10	1	2	5	3	4	15	16	17	18	0	9	14
7	7	8	6	9	0	1	4	5	3	14	15	13	17	2	10	19
6	6	7	8	1	9	10	3	4	5	13	0	2	16	17	18	12
5	5	3	4	0	6	8	10	1	2	7	12	14	9	15	17	13
4	4	5	3	2	7	6	9	0	1	8	13	12	11	16	15	10
3	3	4	5	6	2	0	1	9	10	12	8	7	15	11	16	18
2	2	0	1	5	3	4	8	6	7	11	9	10	14	12	13	17
1	1	2	0	4	5	3	7	8	6	10	11	9	13	14	12	16
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Figure 3: Table of Grundy values for WYTHOFF'S GAME, taken from [Niv04]

So, we can recursively find a Grundy value for a particular position. But, conversely, we can also find the set of positions that have a particular Grundy value. This set of positions will be formalized in Definition 4.3:

Definition 4.3 (from [BF90])**.** Let T_j be the sequence of game positions, in the form of ordered pairs, that have the Grundy value of j. This set will be written as $\{(a_0{}^j, b_0{}^j), (a_1{}^j, b_1{}^j), (a_2{}^j, b_2{}^j), ...\}$, where the subscripts indicate the index of the game position in the sequence and the superscripts indicate the Grundy value of the position. We assume that $b_i{}^j \geq a_i{}^j$ for all values of i and j, and, when j is fixed, we will omit the superscripts. Also, let $D_j = \{b_0 - a_0, b_1 - a_1, ...\}$. D_j contains the differences between the a's and b's for the sequence T_j.

We have a recursive algorithm to generate the sequence T_j with all pairs (a, b) such that $\mathcal{G}(a, b) = j$. Suppose that, for a given k, the previous $k-1$ pairs in T_j have already been constructed. Let $p = mex(a_l,\ b_l : 0 \leq l < k)$, and suppose furthermore that the previous sequences T_i with $i < j$ have been constructed up to the point that each T_i already has p in some of its pairs. We will construct the next pair (a_k, b_k) using the following algorithm, known as the Wythoff-Sprague-Grundy (WSG) algorithm:

Theorem 4.1 (WSG Algorithm, from [BF90])**.** *To generate the* k^{th} *element of the sequence* T_j*, do the following steps:*

1. *Set* p *to* $mex(a_l,\ b_l : 0 \leq l < k)$ *and* q *to* $mex(b_l - a_l : 0 \leq l < k)$*.*
2. *If* $(p,\ p{+}q)$ *does not appear in any sequence* T_i *for* $i < j$ *and* $p{+}q$ *does not appear as the second term in any pair in* T_j *constructed so far, then the* k^{th} *element of the sequence* T_j *will be set to* $(p, p+q)$*. Exit.*
3. *Else, replace* q *with the next smallest integer that is not in* D_j*. Return to Step 2.*

The full proof of the validity of this algorithm will not be mentioned in this paper, but it can be found in [BF90]. Here is a more intuitive, but incomplete, explanation. By setting p to $mex(a_l,\ b_l : 0 \leq l < k)$ in Step 1, we fix our new coordinate to lie in the p^{th} column, and then we look for which row number to give the coordinate. We first assign p, the column number, to $mex(a_l,\ b_l : 0 \leq l < k)$ and q, the "diagonal number"[1], to $mex(b_l - a_l : 0 \leq l < k)$ because we do not want our new coordinate to lie in the same column or diagonal as our previous pairs in T_j, respectively. Keep in mind that the positions $(a_l,\ b_l)$ and $(b_l,\ a_l)$ both have the Grundy value of j. If our new coordinate were to lie in the same column or diagonal as the previous pairs, there would then be two positions with a Grundy value of j in the same column/diagonal, which is inherently impossible because of Equation 4. Then, in Step 2, we check if the coordinate already has a smaller Grundy value in its place, by searching for it in earlier T_i's, and we check if the coordinate shares a row with a previous pair in T_j. If either of these conditions are true, we must find a new row number to give our coordinate, and so we move on to Step 3. By reassigning q to the next smallest integer not in D_j, we change the coordinate's diagonal to the next diagonal that is not "taken" by a previous pair in T_j.

In this paper, we will only focus on finding the $\mathcal{N}$ and $\mathcal{P}$ positions, or the positions whose Grundy value is greater than 0 and the positions whose Grundy value is equal to 0, respectively.

We take the algorithm in Theorem 4.1 and simplify it so that we only generate the sequence T_0, or the sequence of $\mathcal{P}$ positions.

Theorem 4.2. *To generate the pairs in the set T_0, do the following steps:*

1. *Append the game position $(0, 0)$ to T_0.*
2. *Initialize the variable k to 1.*
3. *Repeat the following indefinitely:*
    (a) *Set p to $mex(a_l,\ b_l : 0 \leq l < k)$*
    (b) *Insert $(p,\ p + k)$ in T_0.*
    (c) *Increment k by 1.*
    (d) *Loop back to step (a).*

Using this algorithm, we find a way to compute the k^{th} pair in T_0 as follows, assuming that the previous $k - 1$ pairs are already constructed:

$$a_k = mex(a_l,\ b_l : 0 \leq l < k) \quad (5)$$

$$b_k = a_k + k \quad (6)$$

In this simplified form of the WSG algorithm, notice how we subtly remove the step where we check if the position $(p, p + k)$ shares a row with an earlier position in T_0 and, instead, we directly set $(p, p + k)$ to be (a_k, b_k). This is valid because it is guaranteed that, when we construct new pairs, they must automatically be in different rows than the previous pairs. If we write the sequence of a's as $\{a_n\}$ and the sequence of b's as $\{b_n\}$, it suffices to show that the sequence $\{b_n\}$ is strictly increasing.

[1]The "diagonal number" of a coordinate is the number that represents the diagonal that the coordinate lies in. It is found by taking the difference of the second number and the first number.

Proof for $\{a_n\}$ and $\{b_n\}$ being strictly increasing. First, we have to prove that the sequence $\{a_n\}$ is strictly increasing. Proving that $\{a_n\}$ is strictly increasing is equivalent to showing that, for every n, $a_{n+1} > a_n$. So, let's assume the inverse that $a_{n+1} \leq a_n$, and we show through contradiction that this is impossible. The inequality $a_{n+1} \leq a_n$ breaks down into two cases: $a_{n+1} = a_n$ and $a_{n+1} < a_n$. Let $S = \{a_0, b_0, a_1, ..., a_{n-1}, b_{n-1}\}$, and remember that

$$a_n = mex(S) \tag{7}$$

and

$$a_{n+1} = mex(S \cup \{a_n, b_n\}). \tag{8}$$

The first case immediately leads to a contradiction because of Equation 8; a_{n+1} cannot possibly be equal to any of the previous a's or b's by definition. To disprove the second case, we see that, according to Equation 7, the set S must contain all of the numbers less than a_n due to the definition of the *mex* operation. So, if we assume that $a_{n+1} < a_n$, we deduce that $a_{n+1} \in S$. Again, we reach a contradiction since, by Equation 8, a_{n+1} cannot be in S. Thus, we have proven that the sequence $\{a_n\}$ is strictly increasing. Because $b_n = a_n + n$, it follows that $b_n > a_n$ when $n > 0$, and we can see that $\{b_n\}$ must be strictly increasing too.

■

We can observe some other characteristics of the sequences $\{a_n\}$ and $\{b_n\}$. We see that $\{a_n\}$ and $\{b_n\}$ are almost-disjoint sequences by construction, with the exception of $0 \in \{a_n\}$ and $0 \in \{b_n\}$. When we assign a_n to $mex(a_l,\ b_l : 0 \leq l < n)$, we prevent any element in $\{a_n\}$ from being equal to any of the previous elements in $\{a_n\}$ and $\{b_n\}$. Also, when we assign b_n to $a_n + n$, it is guaranteed that $b_n > a_n$ and that b_n is greater than all of the previous a_i's and b_i's due to the fact that $\{a_n\}$ is strictly increasing, $\{b_n\}$ is strictly increasing, and $b_n > a_n$ for $n > 0$, so we know that b_n does not appear earlier as an element in either sequence. We can also show that $\{a_n\}$ and $\{b_n\}$ must contain every positive integer exactly once. By setting a_n to $mex(a_l,\ b_l : 0 \leq l < n)$, we essentially append to $\{a_n\}$ the missing integers that have not been encountered before in the previous pairs, and so the *mex* operation serves as a "catch-all" for missing integers. Furthermore, since $\{a_n\}$ and $\{b_n\}$ are both strictly increasing, there cannot be any duplicate integers within each sequence, and we know that $\{a_n\}$ and $\{b_n\}$ do not share any elements other than 0, so there is exactly one occurrence of a positive integer in either sequence.

We can verify that $\{(a_0, b_0), (a_1, b_1), ...\}$ is indeed the set of all $\mathcal{P}$ positions using the Partition Theorem.

Proof of $\mathcal{P}$ positions, from [Niv04]. Let the set

$$\mathscr{P} = \{(a_0, b_0), (a_1, b_1), ...\} \cup \{(b_0, a_0), (b_1, a_1), ...\},$$

and let $\mathscr{N}$ be the set that contains all the other positions. We will show the following two parts:

1. Every move from a game in $\mathscr{P}$ will lead to a game in $\mathscr{N}$ (or that no move from a game in $\mathscr{P}$ will lead to a game in $\mathscr{P}$).

2. A move from a game in $\mathscr{N}$ will lead to a game in $\mathscr{P}$.

To prove the first part, suppose that we start a game at the $\mathscr{P}$ position (a_k, b_k). By the construction of the pairs $\{(a_0, b_0), (a_1, b_1), ...\}$, it is known that any move from (a_k, b_k) will not reach another position in $\mathscr{P}$.

To prove the second part, suppose we start in an $\mathscr{N}$ position (x, y), with $x \leq y$. Since the sequences $\{a_n\}$ and $\{b_n\}$ contain each non-negative integer at least once, we know that either $x = b_n$ or $x = a_n$. If $x = b_n$ for some n, then we can move to (x, a_n), which is in $\mathscr{P}$. And, if $x = a_n$ for some n, there are two cases:

1. If $y > b_n$, then we can move to $(x, b_n) = (a_n, b_n)$, which is in $\mathscr{P}$.

2. If $x \leq y \leq x+(n-1) = b_n - 1$, then let $m = y - x < n$. The number m represents the diagonal that (x, y) is in, and we can see that the pair $(a_m, b_m) \in \mathscr{P}$ also lies in the same diagonal since $b_m - a_m = m$. Furthermore, (a_m, b_m) comes before (a_n, b_n) in the sequence T_0 because $m < n$, so we know that $a_m < a_n$ and $b_m < b_n$. Therefore, we are able to move along the diagonal from (x, y) to (a_m, b_m), a $\mathscr{P}$ position.

Hence, we have proven that the pairs (a_n, b_n), as described in Equations 5 and 6, are the $\mathcal{P}$ positions.

Now, we use a crucial theorem to derive our final result:

Theorem 4.3 (Beatty's Theorem, from [Niv04])**.** *Let $\alpha, \beta > 1$ be irrational numbers such that $\alpha^{-1} + \beta^{-1} = 1$. Then the sequences $\{\lfloor \alpha n \rfloor\}_{n=1}^{\infty}$ and $\{\lfloor \beta n \rfloor\}_{n=1}^{\infty}$ contain every positive integer exactly once.*

See the proof of this theorem in [Niv04]. This theorem allows us to construct the two sequences $\{a_n\}$ and $\{b_n\}$ by choosing values for the irrationals α and β. Let the sequences $\{a'_n\}$ be $\{\lfloor \alpha n \rfloor\}_{n=1}^{\infty}$ and $\{b'_n\}$ be $\{\lfloor \beta n \rfloor\}_{n=1}^{\infty}$. If we pick arbitrary values for α and β such that $\alpha < \beta$ and such that they satisfy the conditions above, we can see that

$$a'_n = mex(a'_l,\ b'_l : 0 \leq l < n). \tag{9}$$

Otherwise, an integer would either be repeated or missing in the sequences $\{a'_n\}$ and $\{b'_n\}$.

In finding the sequences $\{a_n\}$ and $\{b_n\}$, we also need another condition to be satisfied: $b_n = a_n + n$ for all n. Now, let ϕ be the golden ratio, or $\frac{1+\sqrt{5}}{2}$. If we set α to ϕ and β to ϕ^2, we see that this condition, as well as the condition in Beatty's theorem, is satisfied because of the fact that $\phi^2 = \phi + 1$. To verify that it satisfies the condition in Beatty's theorem, we have

$$\phi^{-1} + (\phi^2)^{-1} = \frac{1}{\phi} + \frac{1}{\phi^2} = \frac{\phi + 1}{\phi^2} = \frac{\phi^2}{\phi^2} = 1$$

We also show that, when $\alpha = \phi$ and $\beta = \phi^2$, $b'_n = a'_n + n$:

$$\begin{aligned} b'_n = \lfloor \phi^2 n \rfloor &= \lfloor \phi n + n \rfloor \\ &= \lfloor \phi n \rfloor + n \\ &= a'_n + n \end{aligned}$$

So, when we assign α to ϕ and β to ϕ^2, we know that $a'_n = mex(a'_l,\ b'_l : 0 \leq l < n)$ and that $b'_n = a'_n + n$. Therefore, we can conclude that $a_n = a'_n$ and $b_n = b'_n$ for all n. In other words, we get that the set of all $\mathcal{P}$ positions is

$$\{(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)\}_{n=0}^{\infty} \cup \{(\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor)\}_{n=0}^{\infty}.$$

■

We have found the $\mathcal{P}$ positions of WYTHOFF'S GAME. So, if we start a game that is an $\mathcal{N}$ position, the winning strategy is to move to a position of the form $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$ or of the form $(\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor)$.

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