CLASSIC IMPARTIAL GAMES

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1. Introduction

A combinatorial game is a game defined by eight general rules, listed below.

- There are two players, usually called Left and Right.
- There are multiple positions that can be reached during play.
- There are rules which specify the moves players can make from certain positions.
- Left and Right alternate turns.
- At all times, both players know the current position and all possible positions in the future.
- There are no chance elements.
- The first player unable to move loses.
- The game will always end after a finite number of moves.

Combinatorial games are called impartial when both players can make the same moves. In this paper, we will dive deeper into the study of two well known impartial combinatorial games: Fibonacci Nim and Wythoff's Game.

2. Fibonacci Nim

Fibonacci Nim is played by two players, who take turns removing coins from a pile of size n. On the first move, a player is not allowed to take all of the coins, and on each subsequent move, the number of coins removed can be any number that is at most twice the number of coins removed during the previous move. The player who takes the last coin wins. [1]

Fibonacci Nim's winning strategy involves a phenomenon known as the Zeckendorf representation of a number.

Definition 2.1. The *Zeckendorf representation* of a number is the number expressed as the sum of Fibonacci numbers, with no two consecutive Fibonacci numbers in the sum. [1]

The Zeckendorf representation of a number can be found by repeatedly subtracting the largest Fibonacci number possible until reaching 0. [1] The proof of how this ensures that no two consecutive Fibonacci numbers are in the representation is below.

Proof. If a is a Fibonacci number, b is the Fibonacci number right after a, and c is the Fibonacci number right after b, then b and a cannot both be in the Zeckendorf representation of a number as they can be replaced by c, because $c = b + a$ due to the definition of Fibonacci numbers. ■

Another quantity involved in the winning strategy is the quota, which we will call q.

Definition 2.2. The *quota* in Fibonacci Nim is the maximum number of coins that can be taken away on the current turn. On the first turn, $q = n - 1$, and on each subsequent turn, q is two times the number of coins taken away on the previous move. [1]

A winning move for a player in Fibonacci Nim requires q to be greater than or equal to the smallest number in the Zeckendorf representation. If the player can remove all the coins, he/she should do so. Otherwise, he/she should remove the number of coins equal to the smallest number in the Zeckendorf representation, which we will call x. If this happens, the opponent will be unable to make a winning move on the the subsequent turn, because the new quota will be less than the smallest number in the new Zeckendorf representation, which we will call z. [1] This is guaranteed because $z > 2x$, the proof of which is below.

Proof. Both x and z were in the old Zeckendorf representation, so were by definition non-consecutive Fibonacci numbers. The closest they could have been is two Fibonacci terms apart. Assuming z is two terms after x and y is the term between them, $z > 2x$ since $x + y > 2x$ and $z = x + y$. Therefore, $z > 2x$ for all z and x satisfying the requirements above. ∎

Since the new quota is $2x$, the opponent will not be able to make a winning move. If the player cannot make a winning move, he/she will lose the game, assuming the opponent plays optimally using the guidelines above. [1]

Example. Left will move first in this example. Let us say $n = 7$. The Zeckendorf representation of 7 is $5 + 2$. Since $q = 6$ and Left cannot take all the coins, she takes the smallest number in the Zeckendorf representation, 2. This leaves 5 coins and $q = 2 * 2 = 4$. Since taking any number ≥ 2 would allow Left to win immediately on her next turn (if Right takes f coins, then the quota next turn is $2f$ and the maximum allowed number of coins taken on the two turns combined is $3f$; $3f > 5$ for any $f \geq 2$, so Left would win immediately as she could take all the remaining coins on her next turn), Right takes 1 coin, leaving 4 coins and $q = 2$. The Zeckendorf representation of 4 is $3 + 1$, so Left takes 1 coin, leaving 3 coins and $q = 2$. Now, Right is guaranteed to lose. If he takes 1 coin, he leaves behind 2 coins and $q = 2$, so Left can take all the remaining coins and win on her next turn. If he takes 2 coins, he leaves 1 coin and $q = 4$, so Left can take the last coin and win on her next turn.

3. Wythoff's Game

Wythoff's Game is played with two players and two piles of counters. The players take turns removing counters from one or both piles and when removing counters from both piles, the numbers of counters removed from each pile must be equal. When one person removes the last counter or counters, he/she wins. [2]

To analyze Wythoff's Game, we define a position in the game to be a pair of integers of the form (n, m), where $n \leq m$ and n and m describe the number of counters in each pile. The game's strategy involves hot positions and cold positions. A player who is about to move from a hot position will win with best play, and a player who is about to move from a cold position will lose with best play. The optimal strategy of Wythoff's game, if possible, is to move from a hot position to a cold position, as this will force the opponent to lose. If the player cannot move to a cold position, he/she will lose, assuming best play from the opponent using the aforementioned strategy. [2]

The following rules recursively categorize positions into hot and cold: [2]

- $(0, 0)$ is a cold position.
- Any position from which a cold position can be reached in one move is hot.
- Any position from which every move results in a hot position is cold.

Example. Any position of the form $(0, m)$ or (m, m) where m is a positive integer is a hot position as they can all reach $(0, 0)$ in one move, which is a cold position, with a single move. [2]

Example. $(1, 2)$ is a cold position as it can only reach $(0, 1)$, $(0, 2)$, and $(1, 1)$ in one move, which are all hot positions. [2]

Cold positions follow a pattern determined by the golden ratio.

Theorem 3.1. *If r is a nonnegative integer and $x = \lfloor \phi r \rfloor$, $(x, x+r)$ is the rth cold position. [3]*

Using this formula, we find that the first few cold positions are $(0, 0)$, $(1, 2)$, $(3, 5)$, $(4, 7)$, $(6, 10)$, and $(8, 13)$. [2]

Example. Let's say that a game starts off with a position of $(5, 5)$, and Left moves first. Since Left wants to move to a cold position, she'll move to $(3, 5)$. Right is unable to move to a cold position from here, so he removes as few counters as possible to make it harder for Left to win on her next turn, and moves to $(3, 4)$. Left then moves to $(1, 2)$, another cold position. It is now impossible for Right to win as he can only move to $(0, 1)$, $(0, 2)$, or $(1, 1)$, all of which will enable Left to move to $(0, 0)$ on her next turn and win the game.

4. Conclusion

Fibonacci Nim and Wythoff's game are two famous impartial combinatorial games. In this paper, we first identified necessary background information, and then introduced and explained winning strategies for both games. Examples on how to apply these methods when playing the games were also given.

References

[1] Various authors. Fibonacci nim. *Wikipedia*, 2021.
[2] Various authors. Wythoff's game. *Wikipedia*, 2021.
[3] Eric Weisstein. Wythoff's game. *Wolfram MathWorld*, Unknown year.