COMBINATORIAL GAME THEORY AND GO ENDGAMES

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1. Introduction

We will use techniques from combinatorial game theory (of which a basic background knowledge is assumed) to analyze endgames of the game Go. In Go, operations like warming and cooling which normally do not invert each other nearly do, which allows us to employ these techniques to effectively study a hot game.

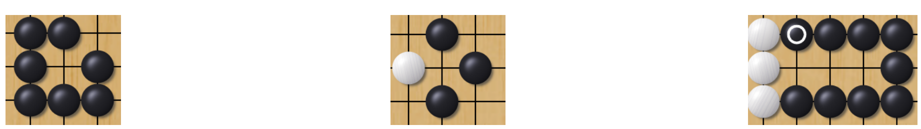
2. The rules

While combinatorial game theory cannot be directly applied to Go, a few simple rules and assumptions will make everything work out.

- Players take turns either placing a stone on the Go board or returning one captured stone to their opponent. Left places black stones; right places white stones. We follow the normal-play convention.
- A stone is captured only if all four vertices adjacent to it are of the other color (Generally we will study subpositions and assume all stones on the edge of the position are connected to groups so they are not captured (alive). This will only not be the case in the example at the end.)
- Once you have placed a stone on a vertex, you cannot place another stone there ever again.
- There is no *seki* (mutual life)

For more details on the rules of Japanese Go, which the above ruleset most closely approximates, see [2].

3. Basic positions



(a) eye **(b)** dame **(c)** blocked corridor

Figure 1. Some simple Go positions

The position shown in figure 1(a), known as an eye, is worth 1 point as black can fill in the middle but white can't. We can expand the space in the middle of the eye and/or add/fuse multiple eyes together to get multiple points. Another simple position is shown in figure 1(b) (there can be whatever on the four open vertices on the outside of the position), and is known as a dame. A dame is essentially an eye but surrounded by stones from both

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players rather than just a single player. Playing in this will leave nothing in this subposition, so it is $\{0 \mid 0\} = *$. While this all seems very nice, a slight change to this position greatly complicates things, as shown in figure 1(c). This is known as a (blocked) corridor. The length of the corridor is the number of vertices inside it, so this one is of length 3. Both players will want to fill from the white side towards the black side (if white puts stones inside the corridor, they will get captured, and black wants to play next to white to take as much space as possible). This means that this game is $\{2 \mid \{1 \mid \{0 \mid 0\}\}\} = \{2 \mid \{1 \mid *\}\}$. This is canonical form, and unfortunately other corridors (and indeed most positions) are similarly (or more) complex. We need more tools to help resolve such positions.

4. Warming, Cooling, and Chilling

We already know what cooling is, and we also know that warming and cooling are not usually inverses. In Go, however, they are very nearly inverses. The warming operation and the cooling by 1, or chilling operation will be instrumental in simplifying Go positions due to the fact that Go tends to be hot. Thus, cooling it down will help it be more amenable to standard combinatorial game theory analysis.

Definition 4.1. A Go game is even if the sum of the number of captured stones and the number of empty vertices on the board is even. Otherwise, it is odd.

Based on the above definition, it is relatively clear that a Go game will alternate between even and odd positions and that the parity of the sum of two Go positions operates the same as the parity of sums of numbers. Note that this excludes a great many positions from being unchilled Go positions (for instance $\frac{1}{2}$ goes to an even position if left plays in it but an odd one if right plays in it, so it is not a Go position).

Definition 4.2. The warming function $\int G$ is defined as

$$\int G = \begin{cases} G & G \text{ is an even integer} \\ G* & G \text{ is an odd integer} \\ \left\{1 + \int G^L \;\middle|\; -1 + \int G^R\right\} & \text{otherwise} \end{cases}$$

Definition 4.3. The chilling function $f(G)$ is defined as

$$f(G) = \begin{cases} n & G \text{ is of the form } n \text{ or } n* \\ \left\{-1 + f(G^L) \;\middle|\; 1 + f(G^R)\right\} & \text{otherwise} \end{cases}$$

Following Berlekamp [1], we will now prove the core of mathematical Go analysis.

Lemma 4.4. *For even game G in canonical form, G and all of its subpositions either have left stop greater than right stop or are of the form n or $n*$ for some integer n.*

Proof. We know $\mathrm{LS}(G) \geq \mathrm{RS}(G)$. Consider the case where $\mathrm{LS}(G) = \mathrm{RS}(G) = k$. If k is even, we must move to k after an even number of moves (all even numbers are even Go positions, and we start with an even position, so there must be an even number of moves from one to the other), so by the number avoidance theorem, $G - k \in \mathcal{P}$ because once G moves to k, it will be the first player's turn to play in $k - k$, which is a $\mathcal{P}$ position. Similarly, if k is odd, there must be an odd number of moves to get to k. In this case, the second player wins the game $G - k*$ because left playing twice in a row in G (pre-stop) will get you a better or equal left stop than k (and similarly for right), so the second player will just play on G until its stop. Once this happens, if the star is still there, the first player was also just playing on G, so the second player takes the $*$ and then the first player loses $k-k$, and if the $*$ is not there, the first player took it at some point, and so there were an even number of moves to reach k (or something better for the second player), so the first player loses the game $k-k$ as well. ■

Lemma 4.5. *$G = \int f(G)$ for even G in canonical form.*

Proof. We will consider a number of cases.

- If $G = n$ (for integer n), $f(G) = n$ and $\int n = n$, so the lemma holds.
- If $G = n* = \{n \mid n\}$, $f(G) = \{-1 + f(n) \mid 1 + f(n)\} = \{n-1 \mid n+1\} = n$. n will be odd because G is even, $*$, and so we have odd+odd = even. As a result, $\int n = n*$, so $\int f(n) = n*$, so the lemma holds.

If G is not of either of the above forms, if $f(G)$ is in canonical form, then the transformation will clearly be inverted. Suppose for the sake of contradiction G has left options A and B such that $f(A) \geq f(B)$. Because of lemma 3.4, the left stops of G exceed its right stops, so the way to play $A - B$ is to not play on n or $n*$ until those are the only options. Let left play $A - B$ moving second. She will just mimic the moves that she made in $f(A) - f(B)$.

- If the number of moves it takes to get to a stopping position in $f(A) - f(B)$ is even, the one point adjustments in the chilled game will cancel when playing $A - B$ and we will get to a stop in the $f(A) - f(B)$ version that is nonnegative, and then since G is even, we can just convert this directly and get $A - B \geq 0$.
- If it takes an odd number of moves to get to a stopping position in $f(A) - f(B)$, the stop must be at least 1 since left won (despite paying an extra one point tax). We can then convert this to the stopping position $*$ on $A - B$, which, since it takes an odd number of moves to reach the stop, left will win. Anything higher than 1 just is better for left. Thus, left wins $A - B$ going second and so G wasn't in canonical form.

We can apply the same reasoning to show $f(G)$ doesn't have reversible options. As a result, $f(G)$ is in canonical form, so the transformation is in fact inverted by $\int$.

■

We now would like to show that chilling and cooling do the same thing in Go so that we can use the properties of cooling, namely linearity, here.

Lemma 4.6. *For $t = \frac{1}{2^k}$ and G a game whose stopping positions (not just left and right stops) are multiples of $2t$, G either has temperature 0 or temperature of at least t. Also, G_t's stopping positions are multiples of t.*

Proof. If G's stops are the same, G is an integer and its temperature is zero. If not, we must cool the two stops until G gets infinitesimally close to its mean, and since the distance between the stops is at least $2t$ we will have to cool by at least t to get them infinitesimally close to each other and thus the mean.

For the second part of the claim, note that all changes to any given part of the game will be multiples of t. This implies that all parts of the game which cool into numbers do so to multiples of t (since all their options either are multiples of t or will cool into multiples of t because all stopping positions are multiples of $2t$), and will then just be further changed by multiples of t. Because of this, we cannot change any stopping position by anything but a multiple of t and they started out as multiples of $2t$, so they will wind up multiples of t. ■

Lemma 4.7. *For an even* Go *game G in canonical form, if G has mean $\frac{i}{2^j}$ for odd i, G's temperature is at least $1-\frac{1}{2^j}$.*

Proof. We will cool G by $\frac{1}{2}$. We can apply lemma 4.6 at each cooling step because stopping positions of unchilled Go games are integers (fractions have some moves leading to odd positions and others to even ones, but the parity of a Go position must alternate) and thus of the form $2(\frac{1}{2})$. This means that $G_{\frac{1}{2}}$ has stopping positions that are mutiples of $\frac{1}{2}$. This in turn allows us to cool by $\frac{1}{4}$ and apply the lemma. We will keep going until we get to $\frac{1}{2^j}$. We have cooled to a total of $1-\frac{1}{2^j}$, so our temperature must be at least this. ■

Lemma 4.8. *For even* Go *game G in canonical form, $f(G) = G_1$.*

Proof. Assume G is a counterexample with minimum birthday. The lemma will holds for all its subpositions. If the temperature of G is greater than 1, either $f(G) = G_1$ or a smaller counterexample is implied. Let G_1 be of the form $\frac{i}{2^j}$ for odd i. Applying lemma 4.7, we know $t \geq 1-\frac{1}{2^j}$. G_t will then have both left stop and right stop of $\frac{i}{2^j}$ (it is the place on the thermograph where the stops meet). Comparing with $f(G)$, $f(G)$ will at most modify the options by another $1-t = \frac{1}{2^j}$, but this would yield $\left\{\frac{i-1}{2^j} \,\middle|\, \frac{i+1}{2^j}\right\} = \frac{i}{2^j} = G_1$. Additionally, just note that there is no way that $f(G) > G_1$ because G_1's left options are less than or equal to those of G_1, so right could win by playing to $f(G) - G_1^L$. ■

Theorem 4.9. *For even* Go *game G in canonical form, $G = \int G_1$.*

Proof. Applying the above lemmas constitutes the proof. ■

Corollary 4.10.

$$\int G_1 = \begin{cases} G & G \textit{ is an even integer} \\ G* & G \textit{ is an odd integer} \end{cases}$$

Proof. This is because if G is even we apply the theorem. If G is odd, $G + *$ is even, so $\int(G+*)_1 = G + *$, but $\int(G+*)_1 = \int(G_1 + *_1) = \int G_1$, so $\int G_1 = G + *$ for odd G. ■

Thus, we have shown a method for chilling/cooling Go games to get them into nice forms and then warming them back up which in this case does actually invert chilling/cooling (note that our proofs relied greatly on the mechanics of Go games - alternating between even and odd positions, having only integer stops which do not apply generally). To employ this strategy, we chill the whole game which is the same as chilling each individual part, cancel where possible, and then make the needed plays to win.

5. General Cases

Theorem 5.1. *An n-length black blocked corridor has chilled value $n-2+2^{1-n}$ for $n \geq 1$.*

Proof. We will prove this inductively. Base case: When $n = 1$, the position is the dame showed earlier. The chilling of $*$ is $0 = 1-2+2^0$, so the theorem holds. Suppose the claim holds for $n = k$. A black corridor of length $k+1$ will have options $\{k \mid G_k\}$ where k is achieved by left playing on the white edge of the corner and thus capturing all but one (the space she played on) vertices of the corridor. G_k is the game for a black corridor of length k, which is what happens when right plays on the vertex touching his intrusion into the corridor, thus shortening its length by one. The chilling of this game is $\{f(k) - 1 \mid f(G_k) + 1\} = \left\{k-1 \,\middle|\, k-1+2^{1-k}\right\}$ using the fact that k is an integer and the induction hypothesis. This is equivalent to $k-1+2^{1-k-1} = k+1-2+2^{1-(k+1)}$, so the theorem holds for the $k+1$ case and thus, by induction, the theorem holds. ■

Theorem 5.2. *A chilled black unblocked (both ends of the corridor have white stones) of length n has value $n-4+2^{3-n}$ for $n \geq 2$.*

Proof. We will prove this inductively. Base case: When $n = 2$, a move by either player will just lead to the dame position, which has unchilled value of $*$. Its chilled value is as a result $0 = 2-4+2^1$, as desired. Suppose the claim holds for $n = k$. A move by left on a corridor of length $k+1$ will lead to a blocked corridor of length k (she plays on one end and blocks it off). A move by right on a corridor of length $k+1$ leads to another unblocked corridor of length k this time (he is going to play next to one of his stones which functionally just shrinks the length of the corridor by one). The chilling of this game is $\{-1+k-2+2^{1-k} \mid 1+k-4+2^{3-k}\}$ using what we know about blocked corridors and the induction hypothesis which simplifies to $\{k-3+2^{1-k} \mid k-3+2^{3-k}\} = k-3+2^{2-k} = k+1-4+2^{3-(k+1)}$, as desired. Thus, by induction, the theorem holds. ■

Note that the negatives of these are clearly true for positions with the colors swapped. It may seem that analysis of most endgames is very simple given that all the general forms we have seen have chilled into numbers. This is not the case. Rather predictably, infinitesimals show up in more complicated positions that are better treated just in a case-by-case basis. In [1], Berlekamp provides theorems for evaluating some more complex configurations, but we will not cover those due to their complexity. Instead, in simple cases, one can merely list out options, chill and work from there.

6. An Example

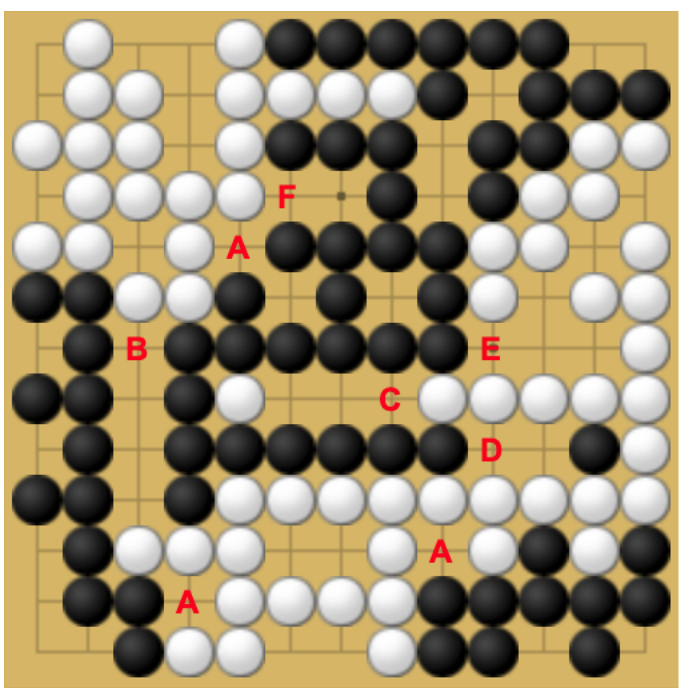

Figure 2. A 13×13 endgame

It is now time to apply the methods we have outlined in theory to an actual example (albeit a somewhat contrived one). The above is a 13×13 endgame that we can apply our methods to.

In this example, right (white) has captured one black stone. Tallying up all the areas that have been completely captured by one player will yield both players having 15 points. Now let's look at the unresolved territory left on the board, each marked where an optimal move would be. There are several dame marked A. These chill to zero. Next, we can recognize B as an unblocked corridor of length 4, so its value is $4 + 2^{3-4} = \frac{1}{2}$. We also recognize F as a length 2 blocked corridor, so its value is $2 - 2 + 2^{1-2} = \frac{1}{2}$. The rest are a bit more complex. In C, A single play by left captures 3 spaces and a white stone for a total value of 4. This value decreases by one for each intrusion right makes, but it will go from 2 (capturing a stone and the space the stone is one) straight to 0. Its options are thus $\{4 \mid \{3 \mid \{2 \mid 0\}\}\}$. Chilling this yields $\{3 \mid \{3 \mid 3*\}\} = 3 + \Uparrow *$. We can apply a similar analysis to D, which has unchilled value $\{\{0 \mid -2\} \mid -3\}$ and a chilled value $\{\{-2 \mid -2\} \mid -2\} = \{-2* \mid -2\} = -2 + \downarrow$. Lastly, we have E. A right play captures 3 vertices of space for himself, whereas if left plays, if right responds, he captures two vertices of space whereas if left plays again, we get two dame, or a zero position. We note that this is $\{\{0 \mid -2\} \mid -3\}$, or the exact same thing as D, so its chilled value is $-2 + \downarrow$.

Overall, we are left with $\frac{1}{2} + \frac{1}{2} + 3 + \Uparrow * -2 + \downarrow - 2 + \downarrow = *$ as our chilled value, warming to get ± 1, and, noting that our game here is odd, adding a $*$ to get our original position. This implies that this is a $\mathcal{N}$ position. The winning move for right here is to take $3 + \Uparrow *$ to $3*$ by playing at C, leaving us with unchilled value $\Downarrow *$ which right will win. To verify, we can warm $\Downarrow *$ which yields $\{1 + \{\{2 \mid 0\} \mid -1\} \mid -1\}$. This is even so there are no changes needed, and one can verify that this is in $\mathcal{R}$. If left goes first, she will want to move one of the $-2 + \downarrow$s to $-2*$, leaving us with a total of $\uparrow$. To verify that left wins this game, we can warm to get $\{1 \mid \{0 \mid -2\}\}$, which is even, so it needs no changes. One can verify that this is in $\mathcal{L}$.

For more examples, see [1] or [3].

References

[1] Elwyn Berlekamp and David Wolfe. *Mathematical Go*. A K Peters, Ltd., Wellesley, MA, 1994.

[2] James Davies. The japanese rules of go, 1989. URL: `http://www.cs.cmu.edu/~wjh/go/rules/Japanese.html`.

[3] Qingyun Wu. Mathematics behind go endgames, 2014. URL: `http://web.stanford.edu/~wqy/research/Honor%20Thesis.pdf`.