THREE PLAYER GAMES

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Abstract. In this article, we discuss principles of two-player game theory that have been extended into three-player combinatorial game theory. We consider restrictive assumptions about the rationales of players that allow for analysis of a three-player game. We investigate examples of impartial and partizan three-player games, such as three-player versions of Nim and Domineering.

1. Introduction

Two-player games always involve two players, L and R that alternate moves in some order. Typically, the first person who is no longer able to make a move loses, and the other player wins, known as the normal play convention. The sole goal of both players is to win and force the other player to lose. If both sides play optimally, one player must always win and the other one must always lose. In partizan games, both players do not necessarily have the same moves available to them at any given moment. In impartial games, the players necessarily have the same moves available to them at any given moment.

Three player games extend the principles of two player games by adding a third player, usually called C for center, who plays in conjunction with Left (L) and Right (R). The convention is that players play in a cyclical order; i.e. $\ldots L, C, R, L, C, R, \ldots$.

We define game options similarly to two-player games. We let G^C be the game options available to C in a game $G = \{G^L \mid G^C \mid G^R\}$.

In contrast, there is no such universal convention for three player games. Efforts to analyze three-player combinatorial game have oftentimes involved restrictive assumptions about the rationales of the players and the formation and behavior of coalitions, where two players of the three players may "team up" in order to defeat the other. These assumptions vary from study to study. Studied examples of Three-Player Games include three-player analogues of Nim, and Domineering.

Many facets of Three Player Games can also be further generalized to games involving any positive integer number $N > 1$ of players [Li, [Li78]].

2. Conventions

The primary idea of three-player games is to extend rulesets of known two-player games to create games where three players play in a cyclical fashion. The three players are customarily called Left, Center, and Right or just L, C, and R. Games have perfect information and no chance. They may be thought of as ordered triples of sets of games previously defined, where each of these sets represents the possible moves of a particular player.

The rule for deciding the winner of a three-player game is not standardized. Usually the last player still able to make a valid move after the other two players have been "eliminated" is declared a winner of the game. However, it is important to note that it is possible that

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there may be games in which no player can force a win. An example of this is a possible game where L has the option to move to a position that is winning for C, or a position that is winning for R. These games are called queer games.

Such games present a challenge for dealing with three-player games. In two-player games, we have a strategy against any opponent if and only if we have a winning strategy against a rational opponent who plays optimally. However, in three-player games, this cannot be guaranteed until we pose certain assumptions on the rationales of the players.

Given any specific three-player game and a player that goes first, we may classify this game into one of 6 categorizations.

Theorem 2.1. *Every finite winner-take-all game of perfect information with players L, C, R can be reduced, uniquely, to a win for L, C, R or to a decision by L, C, R.*

A decision for a particular player implies that the particular player has the option to choose which of the other two players wins, but cannot win himself. In such games, no one player can guarantee victory (Straffin, [Jr.85]).

Thus, once we fix whoever player is going first, our game may be categorized into 6 categories. However, in general there are 11 achievable outcome classes for partizan 3 player games, far more numerous than the 4 possible outcome classes for partizan 2 player games (Cincotti, [Cin05]).

One method of dealing with such games is to assume that the player with the decision will "look back" and see which player "put" him in his undesirable situation. Then, he will let that player lose as "revenge," and the other player will win. This is known as *McCarthy's Revenge Rule* (Straffin, [Jr.85]). This convention applies to games that are known as "winner takes all," where there is one winner and two equal losers.

Another convention has been proposed, known as the *Podium Rule* [Li, [Li78]]. Left, Center, and Right will play cyclically in some order until one player can not move on their turn. At that point, the game is over and the last player to move wins with the second to last player to move coming in second, and the player who was not able to move comes in last. In this way, the players are given a ranking from best to worst, and each player would prefer 1st place to 2nd or 3rd, and would also prefer 2nd to 3rd.

For this reason, each player prefers the player moving right after them to win as opposed to the player moving before them. Namely, Left prefers Center, Center prefers Right, and Right prefers Left. However, an added complication is that if the player immediately after the given player runs out of moves first, then the given player wins. This means that each given player may prefer to sabotage the same player in a given scenario, that they would prefer to win in some other scenario.

3. Partizan Games

First, we begin by analyzing partizan games, where different players have different moves available to them at any given time. We define option notation similarly to two-player games. Just like in two-player games, we let G^L be the set of possible moves available to L, and G^R be the set of possible moves available to R. Furthermore, we also let G_C be set of possible moves available to C. We define the option notation as $G = \{G^L \mid G^C \mid G^R\}$.

Cincotti [Cin05] develops a notion of a number-based notation similar to the numbers used in two-player games. Extending Conway's ideas, a number is defined as a triple of sets of numbers previously defined that satisfy a given set of conditions. In two-player games, a move by each player in a number worsens his/her position. A similar notion of inequality is defined in Cincotti's work that distinguishes numbers from games.

In a normal two-player zero-sum game, the advantage of one player is a disadvantage for the other. In a three-player game the advantage of say, the first player, can be a disadvantage for the second player and an advantage for the third player or vice versa, or perhaps disadvantage for both of the opponents. Cincotti builds his theoretical framework based on this notion of perspective based on each of the three players. Three different relations $\geqslant_L, \geqslant_C, \geqslant_R$ are defined, each representing the subjective point of view of any particular player unrelated to the point of view of the other players.

Definition 3.1. We say that $x \geqslant_L y$ if and only if $y \geqslant_L$ no element in x_C, $y \geqslant_L$ no element in x_R, and no element in y_L is $\geqslant_L x$. We define $\geqslant_C, \geqslant_R$ symmetrically.

We also may define a notion of (disjunctive) game addition.

Definition 3.2. For three-player combinatorial games G and H, $G+H = \{G^L+H, G+H^L \mid G^C+H, G+H^C \mid G^R+H, G+H^R\}$.

The above definition satisfies several desired properties of addition. It is commutative, associative, and satisfies the property that 0 is an additive identity, where 0 is defined as the empty game $\{||\}$.

We recursively define the relation of equality by the following rule:

Definition 3.3. First, we say $G =_L H$ if and only if $G \geqslant_L H_L$ and $H \geqslant_L G_L$. We define $G =_c H$ and $G =_R H$ symmetrically. $G = H$ if and only if $G =_L H$, $G =_c H$, and $G =_R H$.

We may also define the notion of birthday for three-player games. We do this in a similar way as two-player games.

Definition 3.4. The 0 game has birthday 0, and the birthday of a game G is the maximum birthday of its options plus one. We say a game is born by day n if its birthday is less than or equal to n for all nonnegative integers n.

Domineering is a well-known two-player partizan game, and a three-player analogue has also been studied by Cincotti [Cin08]. This game is played on a three-dimensional grid with edges parallel to the x, y, and z axes. Players move by placing three dimensional dominoes with dimensions $2 \times 1 \times 1$. Left (also called X) places dominoes with long sides parallel to the x-axis, Center (also called Y) places dominoes with long sides parallel to the y-axis, and Right (also called Z) places dominoes with long sides parallel to the z-axis. Dominoes are not allowed to overlap and when one of the players cannot find a place for one of its dominoes, he/she leaves the game.

Cincotti proposes a method of determining the winner through a process of elimination. When one player leaves the game, the remaining two players continue in alternation (with the player ordered right after the eliminated player going first) until one of them is unable to move. Then that player leaves the game and the remaining player is crowned the winner. In other words, the game is reduced to a two player game with the normal play convention. This method of determining the winner leaves the possibility open for queer games (in other words, the game is a decision).

In his paper, Cincotti analyzes several positions in 3 dimensional domineering. First, if at least one of the dimensions is fixed as 1, then at least one player will never be able to find a move regardless of who goes first. Then, the game is reduced to an example of a two-player game of domineering in two dimensions. For example, if the x dimension is one, then Left will automatically lose and the game is reduced to a two player domineering game in a two dimensional grid $y \times z$ between Center and Right. In this case, Center will be placing 2×1 dominoes, and Right will be placing 1×2 dominoes. Whoever goes first in this game is determined by the original starting player in the three player game.

Besides these special examples, Cincotti gives an exhaustive analysis of all $x \times y \times z$ boards with $x + y + z \leq 10$ and $x, y, z \geq 2$ over all three possible starting players. In most cases, none of the players are able to force a win (in other words, the game is a decision).

Furthermore, Cincotti derives several theorems relating to three-dimensional domineering positions $2 \times 2 \times z$ be a three-dimensional board where z is even and greater than or equal to 4.

Theorem 3.5. *Let $2 \times 2 \times z$ be a three-dimensional board with z even and greater than or equal to 4. We have the following:*

(1) *If Left starts the game, he cannot have a winning strategy.*
(2) *If Center starts the game, he cannot have a winning strategy.*
(3) *If Left starts the game, Center does not have a winning strategy.*
(4) *If Center starts the game, Left does not have a winning strategy.*
(5) *If Right starts the game, neither Left nor Center has a winning strategy.*

This implies the following:

Corollary 3.6. *Let $2 \times 2 \times z$ be a three-dimensional board with z even and greater than or equal to 4. Either Z is winning or the game is queer.*

4. Nim

Now we analyze impartial games, where different players will always have the same moves available to each of them at any given time.

Much of the research done on three-player impartial games has been done on the three-player analogue of Nim. In Nim, the three players take turns moving objects from distinct heaps or piles. On each turn, a player must remove at least one object, and may remove any number of objects provided they all come from the same heap or pile. When a player cannot move any longer the game ends. Typically the last player to win is considered the winner.

Propp's research [Pro00] focuses precisely on those possibilities where one player has the ability to force a win. He proposes four possible outcome classes for a three-player impartial game. N for a next player win, O for an other player (or second player) win, P for a previous player win, and Q for a queer game, where the winner cannot be determined unless we make an assumption about the rationales of the players. He gives a recursive method for determining the outcome class of any impartial game:

(1) $G \in N$ if it has a P option.
(2) $G \in O$ if it has an option and all options are in N.
(3) $G \in P$ if all its options are in O.
(4) $G \in Q$ if none of these conditions are satisfied.

The simplest Nim game is the null-game 0, which is the unique game with no available options. This is classified as P. From there, Propp goes on to classify several other simple Nim games and analyze their outcome classes.

First, we have all single pile games as N games, because in each case Next can win by taking the entire heap.

For notational convenience, let $GH = G{+}H$, $G^n = \overbrace{G + G + \cdots + G}^{\text{n times}}$, and $m_n = \overbrace{\{\{\ldots\{m\}\ldots\}\}}^{\text{n layers deep}}$. Then, we have the following:

(1) $1 = \{0\} = \{P\} = N$
(2) $11 = \{1\} = \{N\} = O$
(3) $111 = \{11\} = \{N\} = P$

and so on; in general 1^n is P if $n \equiv 0 \pmod 3$, N if $n \equiv 1 \pmod 3$, and O if $n \equiv 2 \pmod 3$. An easy extension of this is that the type of 1^n2 is N, Q, or N corresponding whether to the residue of n mod 3 is $0, 1$, or 2 respectively. The winning strategy is to reduce the game to one of the 1^n positions.

Propp also proposes the following general Three-Player Nim addition table.

+	P	N	O	Q
P	PQ	NQ	OQ	Q
N	NQ	NOQ	PNQ	NQ
O	OQ	PNQ	NQ	NOQ
Q	Q	NQ	NOQ	OQ

Finally, Propp proposes a set of equivalence classes encompassing all Three-player Nim games. He proves that every Nim game can be shown to be equivalent to one of a small set of Nim games that he calls *reduced* Nim games. He provides a algorithm of how one may convert any given Nim game into its reduced form.

We move onto research done by Li in a general study of N-player Nim and N-player Moore's games. Li adopts the podium rule in order to avoid queer games. We number the n players $P_1, P_2, P_3, \ldots P_n$ in the order that the play initially. The players play cyclically until one player, P_m, cannot move on their turn. The players are given a ranking in the order $P_{m-1}, P_{m-2}, \ldots P_1, P_n, P_{n-1}, \ldots P_m$. In particular, the last person that made a move is the winner, and the person who couldn't make a move is the loser. Each player seeks to have the highest rank possible. The rank of one player determines everyone else's rank. We then classify the outcome of the game based on what happens when each player adopts an optimal strategy toward his own highest possible rank.

If the mth person to move is the biggest loser, Li calls it an $m-1$ position. For a player to achieve his highest possible rank, he should aim to move into a position with the smallest number possible.

Li proves the following theorem that allows us to find all P positions in N-player Nim with the podium constraint.

Theorem 4.1. *Consider the n-person Nim game of h heaps of sizes $c_1, c_2, \ldots c_h$, respectively. Express the c's in binary notation and add them together without carrying and in the scale of n (as in carrying occurs modulo n). Then the resulting number is a* 0 *position if and only if the game is a* 0 *position.*

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