Surreal Numbers

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Surreal Numbers (**No**) are a proper class of numbers that includes the real and ordinal numbers. They're defined inductively, each as an ordered pair of sets of previously defined numbers, such that all elements of the first set are less than all elements of the second. Each rung on the inductive ladder is called a birthday to give a way to refer to when each one is created. On day zero, the first element, "0" is born. Since there are no prior elements, it is defined on the empty set as $(\{\},\{\})$ or as it's more commonly written: $\{\{\}|\{\}\}$ or just $\{|\}$. Notice how all elements in the set on the right are greater than each element in the set on the left, because there are none of either, thus it is "vacuously true." On day 1 we can create two numbers which satisfy our conditions: $\{\{|\}|\}$, or $\{0|\}$ and $\{|0\}$, which, once we define arithmetic, will have the useful names of 1 and -1 respectively. Notice how we can't create $\{0|0\}$ because $0 \not< 0$ (though interesting things happen if we do allow it). On day 2, we get $2 = \{1|\}$ and $-2 = \{|-1\}$. $\{1|0\}$ and $\{0|1\}$ also show up which turn out to be $\frac{1}{2}$ and $\frac{-1}{2}$ respectively, though that will take a bit more work. We are also able to show that the numbers like $\{-1|1\}, \{0,1|\}$, and $\{|0,-1\}$ aren't actually new numbers (0, 2, and -2 respectively).

When talking about these numbers, we normally use $x = \{x^L|x^R\}$, from the use of surreal numbers in game theory, where x^L represents the set of left options of x. That is, in a combinatorial game, the moves the player named "left" would be allowed to take. Similarly for x^R, the right options of x. For example, if this were tick-tack-toe, one set would have all the board positions after a new X was drawn, the other would have them after an O was drawn. Each of these subpositions could be written as simply the board itself, or in the above form of all the possible next moves recursively.

We're only going to go about defining addition here, but more information on these definitions can be found in Simon's book or On Numbers and Games or many other places. (spoiler alert: it neatly forms a Field)

We'll define $x+y = \{x^L+y, x+y^L|x^R+y, x+y^R\}$ where x^L represents *all* the left options of x, if there were multiple, we would have all of $x^{L_1}+y, x^{L_2}+y, ...$ as left options of $x+y$. Similarly for x^R, y^L, and y^R. In the game theory origin, this represents "conjoined" games where you can make one move in your choice of the games, like if you were playing two chess games simultaneously where each turn you would choose which board to make your move on. Notice how if you, left, made a move in the board x, the new position would be $x^{L_n}+y$, with x^{L_n} representing the new board after you made your move, and the $+y$

because the second board remained unchanged. This is why we keep track of where both players can move on any turn, even if in many typical games players would always be alternating. Multiplication doesn't have as simple a foundation in game theory, but they are defined to give the same answers as the same equation with real numbers.

At this point though, the induction doesn't seem particularly interesting. We have the integers and dyadic rationals (the fractions $\frac{p}{2^q}$ where p and q are integers.) but this is far from the promised reals and ordinals (and more!). On the whole number birthday n, the largest number was born that day, $\{n-1|\} = n$, but on birthday ω, the first transfinite ordinal, we get $\{0, 1, 2, ...|\}$ which must be larger then every integer, hence this number must be ω itself! This pattern continues with $\{1, 2, 3, ..., \omega|\} = \omega + 1$ on birthday $\omega + 1$, and even $\{\omega + 1, \omega + 2, \omega + 3, ...|\} = \omega^2$ on birthday ω^2!

The birthday ω has even more significance though, as it is also the day we get every other rational and every irrational! for example: $\frac{1}{3} = \{0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, ...|1, \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, ...\}$ with dyadic rationals approaching it from either side. This strategy works for every irrational as well: $\pi = \{\lfloor\pi\rfloor, \frac{\lfloor 2\pi\rfloor}{2}, \frac{\lfloor 4\pi\rfloor}{4}, \frac{\lfloor 8\pi\rfloor}{8}, ...|\lceil\pi\rceil, \frac{\lceil 2\pi\rceil}{2}, \frac{\lceil 4\pi\rceil}{4}, \frac{\lceil 8\pi\rceil}{8}, ...\} = \{3, 3.125, 3.140625, ...|4, 3.5, 3.1875, 3.15625, ...\}$. Since this goes on for ω, both these sequences approach, and so to speak "finally reach" π, getting closer than arbitrarily close. It's rather interesting that this construction doesn't draw attention to the difference between non-dyadic rationals and irrationals, which are normally completely different ballparks.

An interesting property of the surreal numbers is their lack of continuity or completeness. We have created the notion of completeness to distinguish between numbers like the discrete integers and the real numbers. Perhaps unintuitively, the rational numbers are also incomplete, despite being being arbitrarily close to each other. One way to demonstrate this is with Dedekind sections, a manner to point out the "gap" in the middle, if one exists. We construct two sets (or classes) of numbers, such that none in the left set are equal to or larger than any element of the right set, and every number is in one of them. If neither has a closed bound (the left set doesn't have a maximal element and the right set doesn't have a minimal element) we say there's a gap. Hopefully an example will help this make more sense:

In the rational numbers we can demonstrate the gap of $\sqrt{2}$ by creating this Dedekind cut:

$$\{\{l|l^2 < 2\}|\{r|r^2 \geq 2\}\}$$

Of course the left set never has an upper bound, but for $r \in \mathbb{Q}$, neither does the right, thus this is a gap in the rational numbers. If we wished, we could name every gap in the rationals to extend the set, which would be a valid construction of the real numbers.

Now we may preform a similar strategy on the surreal numbers, though we are required to make a clarification that these cuts must be with classes rather than sets. A class is a collection of sets, defined by a formula whose quantifiers range over sets, which allows us to rigorously refer to collections that wouldn't themselves be sets. This is crucial for the surreal numbers because we recognize that in our inductive construction, we defined a new number for every set of numbers already existing. Thus for every set of surreal numbers, finite or not, there exists a surreal number that is not in that set, hence the notion of "the set of surreal numbers" doesn't really have meaning. Classes can be used to circumvent this problem. By the definition of a class though, we'll also notice that many infinite and all finite classes of sets are also sets of sets, thus we create the notion of a Proper Class: a class which is not a set. Similarly, the common notion of arithmetic as previously defined does not form a field, which requires its elements to form a set. Thus we defined Fields, Groups, etc. to be all those same objects except defined over Proper Classes.

When using Dedekind cuts on the rational numbers, the requirement that both sets are non-empty is often included, but from the precedent of Conway, this is not the case with the surreal numbers. Conway names the gaps $\{\mathbf{No}|\}$ and $\{|\mathbf{No}\}$ **On** and **Off** respectively. For those unfamiliar with game theory, the games **On** and **Off** represent an unbeatable advantage. Using option notation, we can express these as $\mathbf{On} = \{\mathbf{On}|\}$. Remember, these are games where you lose when you run out of moves, and here they can move as many times as they want without changing their position. Expressing these games in option notation also shows why they're not numbers, there's no birthday that would have **On** or **Off**, because their construction requires themselves to already be born.

We can also demonstrate the gap between the largest real numbers and the smallest trans-finites, where the left set contains all the numbers less than 1, 2, 3,... and the right set contains all numbers more than $\omega, \omega-1, \omega-2, ...$ Since their is neither a largest real number, nor a smallest transfinite, and no numbers between them, we have another gap. Many more trivial gaps can be found between real numbers and the infinitesimals around them, and translations of these across the number line,

These gaps may seem much more arbitrary than those in the rational numbers, but that's mostly because we're already intimately familiar with the reals. It turns out there are actually games in these gaps in the number line. For those familiar with games this shouldn't be too surprising as they are defined very similarly to the surreals, except without the left set$\leq$right set clause. (and in some cases, more exceptions). To list a few examples, the tiny games happily slot between 0 and the smallest "ordinal-reciprocal" infinitesimals. (and 1+tiny between 1 and those infinitesimals +1, etc.) As said before, **On**, the game where left my always move back to **On** and right has no moves, also fits in the gap past every number. Unfortunately though, simply removing this criterion is far from a clean extension, as many other games show up, including * or $\{0|0\}$ which is incomparable to 0. Despite looking mostly unrelated to games, a the surreal numbers can form a Dedekind extension the normal way.

Deeper concepts can be found in Simon and A. Swaminathan's paper Analysis on Surreal Numbers, where they outline a normal form of gaps in **No** iterated over the ordinals. The paper goes a lot further, building a topology on **No** (which is a little beyond me...) and using it to define and demonstrate crucial parts of calculus even with the lack of completeness.