MANCALA-LIKE GAMES

EDDIE FUJIMORI

1. Introduction

Originally created in Egypt, Mancala-like games have been observed in hundreds of different formats throughout the world since. Different from other types of games like "Taking and Breaking","Cutting and Coloring", and "Sliding and Jumping" type games, Mancala-like Games follow a general style of game called "sowing".
Proposed as a type of game by John Conway and Richard Guy based off of the original African games, "Sowing Games" consist of games with pots where players alternate putting stones or seeds into successive pots. Additionally, different from many classic combinatorial games, many sowing games involve winning by points rather than specific win conditions. By working on these types of sowing games, Conway and Guy believed that it would effectively provide a more solid and thorough understanding of combinatorial game theory.
Aside from generally following the pattern of sowing games coined by Conway and Guy, Mancala-like games, because of its pieces and the winning condition of points, rarely rely in any way on luck but purely on skill and strategy. Additionally, in terms of cultural and religious importance, it is the source of copious amounts of superstition and rituals stemming throughout different regions in Africa.

2. Rules of the Game

The rules of the game Sowing shares the most similarities to the various types of sowing games. Initially let there be some number of pots each containing some arbitrary quantity of seeds. Left may take all the seeds out of any pot and sow them to the right under the condition that the last seed can't go into an empty pot. The same with Right except they must sow to the Left. This is standard play. However, we can also consider the impartial variation where Left and Right can move both directions.

3. Sowing Positions

Sowing positions are represented as a combination of bolded numbers which represent the amount of seeds in each pot. For example, from the position 312, Left's only legal move is to move the single seed in the second pot into the third pot, leaving the position 303. Right has two legal moves, to 402 and 420. Thus,this position can be evaluated as follows using option notation.

$$312 = \{303|402, 420\}$$
$$= \{0|\{|510\}, 0\}$$
$$= \{0|\{|\{|600\}\}, 0\}$$
$$= \{0| - 2\}$$

. cited from [Eri94].

4. Simplifying Positions

Definition 4.1. We will call a pot full, represented by the symbol (•), if the number of seeds is greater than the distance to either the first or last (nonempty) pot. The specific amount of the seeds in a full pot doesn't matter.

This somewhat simplifies certain strings of pots and seeds because if some amount of pots exceed the maximum number, which as mentioned before doesn't matter, then we only have to consider the other pots. However, seeds or stones can still be dropped into full pots so unlike empty pots they cannot be ignored.

5. Time Complexity

Theorem 5.1. *Evaluating Sowing positions by recursively evaluating all their followers requires exponential time in the worst case.*

It is possible to evaluate the sowing positions faster, although analysation of higher-level patterns is needed.
To further emphasize the difficulty of finding the time it takes and its length, we can consider a position of the game "Sowing" called Towers of Hanoi.

Towers of Hanoi

In the Towers of Hanoi, let there be n pots with 1 seed. We wish to move all the seeds to the last pot.
Jeff Erickson's paper [Eri94] describes a thorough algorithm that recursively solves the problem though it's not the fastest.

- If $n = 2m$:
  - Move recursively to $m0^{m-1}m$.
  - Sow the first pot to $1^{m-1}(m+1)$.
  - Move recursively to $(2m)$.
- If $n = 2m+1$:
  - Move recursively to $m10^{m-1}m$.
  - Move to $(m+1)0^m m$.
  - Sow the first pot to $1^m(m+1)$.
  - Move recursively to $(2m+1)$.

The number of moves $T(n)$ used by this algorithm obeys the following recurrence:

$$T(1) = 0, \quad (1)$$

$$T(2m) = 3T(m) + 1 \quad (2)$$

$$T(2m+1) = 2T(m) + T(m+1) + 2. \quad (3)$$

So, $T(n) = O(nlog_2 3) = O(n1.5850)$. This yields us the final result of $T(2^k) = \frac{1}{2}(3^k - 1)$.

6. Sowing Patterns

Generally, we know that there are Sowing positions whose values are arbitrary integers and switches with arbitrarily high temperatures. Besides this, much of sowing patterns is unknown or near impossible to determine [Now98]

Theorem 6.1. $(10)^m03(01)^n = 0$ *for all* m *and* n

Proof. If Left goes first, she loses immediately, since she has no legal moves. If Right goes first, his only legal move is to the position $(10)^{m-1}211(01)^n$, from which Left can move to $(10)^{m-1}0220(01)^n$, which has value zero since there are no more legal moves. Thus, the second player always wins. □

Theorem 6.2. $(01)^m2(01)^n = n+1$*, for all* m *and* n *except* $m = n = 0$*.*

Proof. Right has no legal moves. If $n = 0$, Left has only one legal move, to $(10)^{m-1}03 = 0$ by the previous theorem. Otherwise, Left has exactly two legal moves, to $(01)^{m+1}2(01)^{n-1} = n$ by induction, and to $(10)^{m-1}3(01)n = 0$, which is a terminal position. □

Theorem 6.3. $11(01)^n = \{n+1|0\}$ *for all positive* n*.*

Proof. Left has only one move, to $2(01)^n = n+1$ by the previous theorem. Right has only one move, to the terminal position $20(01)^n$. □

Theorem 6.4. $(10)^m2(01)^n = \{n|-m\}$ *for all positive* m *and* n*.*

Proof. Left has only one move, to $(10)^m012(01)^{n-1}$, which, by a slight generalization of Theorem 3, has value n. Similarly, Right has only one move, to $(10)^{m-1}210(01)^n = m$. □

Winning Positions for Stones

Theorem 6.5. *Discussed and proven by Richard Nowakowski [Now98], given an infinite number of pits there will be a winning position for any number of seeds or stones. Given finitely many pits, this is not necessarily true or guaranteed.*

For proof refer to [Now98].

7. Chinese Remainder Theorem

Aside from analysing these sowing games by themselves, we can also use topics from other areas of math or compare common patterns. An example involves a Mancala-like game called “Tchoukaillon”. [JTT13] was able to create an analogue and compare this game to the Chinese Remainder Theorem.

8. Relating Mancala-like games

Although the previously discussed theorems and definitions usually only apply to a few games, most sowing games follow a similar gist so we can easily relate them with one another. A common example is with a game called “Ayo” and a game called “Tchoukaillon”. Both games involve the pits and a larger pit towards the end. However, although “Tchoukaillon” has dropped off over the years, one involves 2 rows and one involves 1. If we observe a row of a game of “Ayo” numbered $1, 2, ..., n+1$ it would be identical to that of a game of “Tchoukaillon”. [Now98].

References

[Eri94] Jeff Erickson. Sowing games. In *Games of No Chance, Proc. MSRI Workshop on Combinatorial Games*, pages 287–297, 1994.

[JTT13] Brant Jones, Laura Taalman, and Anthony Tongen. Solitaire mancala games and the chinese remainder theorem. *The American Mathematical Monthly*, 120(8):706–724, 2013, https://www.tandfonline.com/doi/pdf/10.4169/amer.math.monthly.120.08.706.

[Now98] Richard J Nowakowski. *Games of no chance*, volume 29. Cambridge University Press, 1998.