Mancala–Like Games

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1 Introduction

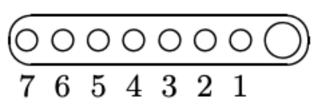
Mancala is a strategy–based two player game with holes and seeds. It's origins are unknown due to Mancala's old age, but it's believed that it originated from Africa from evidence dating back to 3,600 years ago. Mancala is called a *sowing game* because of the way that players plant seeds in holes.

There are currently more than 800 known varieties of Mancala, but we will only study the winning positions between the two renditions of Mancala: Tchoukaillon and Ayo.

1.1 Rules

1.1.1 Tchoukaillon

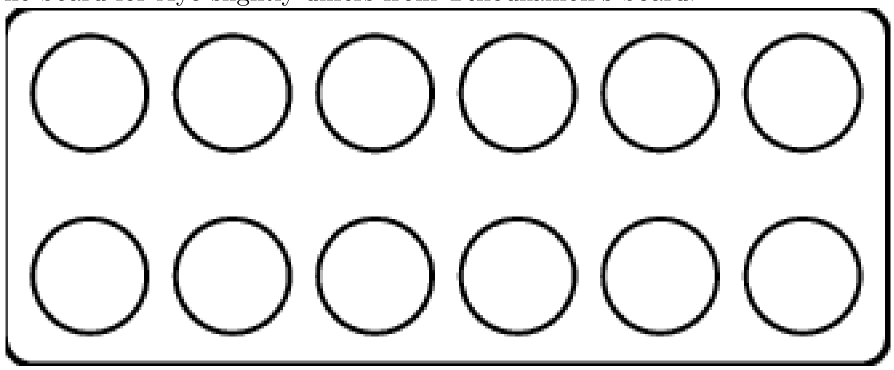
Tchoukaillon is played with some amount of small pits and one large pit at the end which is sometimes called the *Roumba*.



The rules for a one–player game of Tchoukaillon are as follows:
Setup: Any number of seeds, or small objects, are placed into each of the pits other than the Roumba.
Objective: Collect all seeds in the Roumba
Turn: During a turn, the player can choose any one of the pits and remove all the seeds in it. Then, the player redistributes one seed into each of the pits that follow into the direction of the Roumba. A pit cannot be harvested if the last seed does not end up in a pit.
End of Game: The game ends when either a player places a seed into an empty pit or when a player collects all seeds in the Roumba. If a player places a seed into an empty pit, the player loses. However, if a player collects all seeds in the Roumba, the player wins.
Two-Player: The two-player version of Tchoukaillon is slightly different from the one-player version. If played with two players, each player plays until they lose and whoever has collected the most seeds in the Roumba wins.

1.1.2 Ayo

: The board for Ayo slightly differs from Tchoukaillon's board.



Next, the rules for Ayo are as follows:
Setup: 48 seeds are distributed into 12 pits so that each pit contains 4 seeds. Each player owns a side.
Objective:Collect the most seeds
Turn:A player chooses any one of the pits on his or her side and redistributes, or "sows" one seed into each of the following pits in a counterclockwise direction. If a player sows his or her last seed on the opponent's side and now contains 2 or 3 seeds in the pit, then those seeds are captured by that player. Any of the seeds in the preceding pits that have been sown with 2 or 3 seeds are also captured. If a player cannot move in a way that gives an opponent a legal move, then the player captures all of the seeds on the board. If both players have legal moves, but the remaining seeds can never be captured, then each player captures the seeds on their side.
End of Game The game ends when there are no more seeds to capture.

1.2 Winning in Tchoukaillon

In one player Tchoukaillon, not all boards are winnable. Because Tchoukaillon is dictated by strict rules, only few boards are actually solvable. However, there is a way to determine a Tchoukaillon board's solvability.

Theorem 1 *Say that the sequence of integers $(a_1, a_2, ..., a_k)$ is the amount of seeds in each pit from left to right, so that the leftmost pit has a_1 seeds, etc. Then, the board $(a_1, a_2, ..., a_k)$ is solvable. if $a_i \leq i$ for all i and $\sum_{i=j}^{k} a_j \equiv 0$(mod i)*

Proof: First, a Tchoukaillon board should only be solvable if $a_i \leq i$ for all i because a player must place his or her last seed in the Roumba. Thus, if a player distributes $a_i > i$ seeds into i pits, then the last seed will not end up in a pit. Playing any other pit j such that $j > i$ would only result in a_i growing and playing any pit l such that $l > i$ would result in $a_i > i$. This means that if $a_i > i$ for any pit i, the person playing the Tchoukaillon board will never be

able to harvest the seeds in pit i and consequently never win. Next, we must prove $\sum_{j=1}^{i} \mathrm{b}_j(n) \equiv n(\mathrm{mod}\ i+1)$,
where $b_j(n)$ represents the number of seeds in pit j of a solvable Tchoukaillon board with n total stones. We prove this by induction, so we assume the congruence is true. Finding $b(l+1)$ from $b(l)$, we get:

$\sum_{j=1}^{i} \mathrm{b}_j(n+1) = (\sum_{j=1}^{i} \mathrm{b}_j(n))+1$

This is also congruent to $(n+1) mod (i+1)$ since

$\sum_{j=1}^{i} \mathrm{b}_j(n) \equiv n(\mathrm{mod}\ i+1)$.

Therefore, by induction,

$\sum_{j=1}^{i} \mathrm{b}_j(n) \equiv n(\mathrm{mod}\ i+1)$

for any game b(n)

Now that we have proved this, let our original sequence satisfy $\sum_{i=1}^{k} \mathrm{a}_i = n$. Then subtracting our initial congruence $\sum_{i=j}^{k} \mathrm{a}_j \equiv 0(\mathrm{mod\ i})$
from $\sum_{i=1}^{k} \mathrm{a}_i = n$ gives $\sum_{i=1}^{j} \mathrm{a}_i \equiv \mathrm{n}(\mathrm{mod\ i})$.
So, our theorem holds true for solvable Tchoukillion boards.

In addition to determining if a board is winnable, we can also determine the minimum amount of total seeds needed for a board of length l. Let us denote the minimum number of stones as $S(l)$.

Theorem 2 $S(l) = \frac{2}{1} \left\lceil \frac{3}{2} \ldotp\ldotp \left\lceil \frac{l-1}{l-2} \left\lceil \frac{l}{l-1} \right\rceil \right\rceil ... \right\rceil\rceil$

Proof:By Theorem 1, let $a_l = l$ and let us choose a_i and a solvable board of length l with minimal seeds such that
$a_i + \sum_{j=i+1}^{l} \mathrm{a}_j \geq \sum_{j=i+1}^{l} \mathrm{a}_j$
Assuming that
$\sum_{j=i+1}^{l} \mathrm{a}_j = (i+1) \left\lceil \frac{i+2}{i+1} \ldotp\ldotp \left\lceil \frac{l-1}{l-2} \left\lceil \frac{l}{l-1} \right\rceil \right\rceil ... \right\rceil\rceil$
and if $\sum_{j=i+1}^{l} \mathrm{a}_j$ is the minimal amount of seeds needed for a solvable Tchoukaillon board of length l, then a_i must ensure
$\sum_{j=i}^{l} \mathrm{a}_j = (i) \left\lceil \frac{i+1}{i} \ldotp\ldotp \left\lceil \frac{l-1}{l-2} \left\lceil \frac{l}{l-1} \right\rceil \right\rceil ... \right\rceil\rceil$
can also be written as the minimal amount of seeds needed for a solvable Tchoukaillon board of length l. We can continue with induction.

Lastly, if a particular Tchoukaillon board is actually solvable, then there is a simple strategy to win.

Theorem 3 *If a Tchoukaillon board is solvable, then the winning move is to harvest the pit with the least number of seeds.*

Proof: Suppose we have a Tchoukaillon board $(a_1, a_2, ..., a_n)$ and pits i and j where $i < j$. If we were to harvest pit j, then the number of seeds in pit i would increase, and eventually, a_i would grow larger than i, which would cause pit i to never be harvestable. Hence, our best move is to harvest the smallest harvestable pit.

In addition to this strategy, we can also determine the number of times a pit will be played.

Definition: Let $(m_1, m_2, ..., m_k)$ be a move vector that corresponds to the pits 1 to k.

:

Theorem 4 *The move vector can be recursively defined:*

$$f(x) = \begin{cases} a, & if\ i = l \\ \frac{1}{i}(a_i + \sum_{j=i+1}^{l} m_j), & if\ \ 1 \leq i \leq l-1 \end{cases}$$

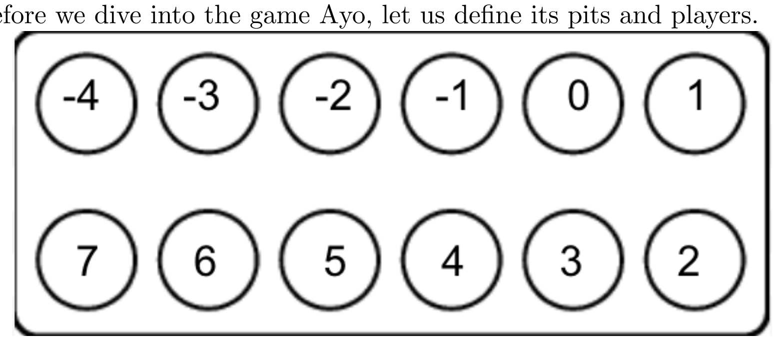
Proof: Let $(a_1, a_2, ..., a_l)$ be a winning Tchoukaillon board, where a_i is the number of seeds in each pit i. By Theorem 1, the last bin must have l seeds so that $a_l = l$. This means that pit l must be played only once, which means $m_l = 1$. Next, we notice that each play that from bin i involves picking up i seeds in total (pit i to Roumba). This means that if pit i, where $1 \leq i \leq l-1$, is played m_i times, then pit i must be involved in picking up im_i seeds in the whole duration of the game. We can also write the total amount of seeds that bin i handles as the sum of a_i and $(m_{i+1}) + m_{i+2} + ... + m_l)$. We can do this because the number of total seeds that bin i moves during the game is also the number of seeds that pit i possessed originally and the number of seeds pit i picks up, which is the same as the amount of times pits $i+1$ to l are played. Thus,

$$a_i + \sum_{j=i+1}^{l} \mathrm{m}_j = \mathrm{im}_i$$

Dividing both sides by i gives us our result for our piecewise function.

1.3 Ayo

Despite the fact that Ayo seems significantly different from its family member Tchoukaillon, they share a few similarities. One of which is their positions. Before we dive into the game Ayo, let us define its pits and players.



Let us define R as right player and $Left$ as left player. Additionally, let the top side be left's side and the bottom side be right's side.

Definition: A determined position in Ayo is a board such that its seed distribution allows R to capture at every turn, L has only one seed on his or her side, and all stones except for one are captured by R

Now, we can move onto the theorems.

Theorem 5 *The seed on L's side of a determined position in Ayo must be in pit 1 if it is L's turn or in pit 0 if it is R's turn.*

Proof: During R's turn, R must capture every seed other than 1. To ensure this, we must have only one seed on L's side that can be captured. Since each pit can only have no more than 4 seeds, this seed may only lie in pits -2 to 1 in order to ensure that R can reach this pit. However, if R plays into -2 or -1, R would capture the 2 seeds in either -2 or -1, but it would give L a chance to capture 2 seeds, which cannot happen in a determined Ayo position. Now we know that the single seed on L's side can only reside in pits 1 or 0. With this, we must now determine that the seed will be in pit 1 on L's turn and pit 0 on R's turn. Assume that L has his or her single seed in pit 0 during his or her turn. Then L can simply move into pit -1 and L will capture 2 seeds in the next turn, which cannot happen. Additionally, if the seed is in pit 1 on R's turn, then R will end up capturing all seeds or leaving enough seeds in the other pits to let L capture more than 1 seed, which cannot happen. However, if we check the game where the seed is in pit 0 on R's turn and pit 1 on L's turn, then we see that the seed alternates between pits 0 and 1 and all seeds except 1 are captured.

Theorem 6 *There is a bijection between determined Ayo positions and winnable Tchoukaillon positions. We can map these Ayo positions to Tchoukaillon boards by ignoring the pits* $0, -1, -2, ..., -n+2$ *in the Ayo board.*

Proof:Let A be a determined Ayo position and T be a winnable Tchoukaillon position such that A and T correspond to each other. This also means that we get the game T when we ignore pits $0, -1, -2, ..., -n+2)$ from the game A. First, we prove that the moves in A correspond to the moves in T. If pit 1 has a stone in the Ayo board, then the seed will have to go to pit 0 by Theorem 5. A similar move can be legally performed in Tchoukaillon by removing a seed from pit 1. However, if pit 1 is empty, then by Theorem 5, R must capture the seed in pit 0 by harvesting a pit n with n stones. A similar move can be legally performed in Tchoukaillon by removing i seeds from pit i and distributing the seeds to score a point in the Roumba. Thus, all moves from A can be replayed in a similar way in T . Next, we notice that at the end of a determined game A, the board is empty, much like at the end of a winning Tchoukaillon game. In conclusion, we notice that corresponding games T and A are both winning positions, their moves correspond, and we also always know the seed's location on R's side of the board for A by Theorem 5. Because we always know the seed's location on R's side in game A, we can determine the corresponding game T from A by ignoring pits $0, -1, -2, ..., -n+2$.

References

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