CGT Paper

Chris Bao

July 2021

1 Introduction

In this paper, we will introduce bidding games, and derive some results about them. A Bidding Game (note: come up with a more creative name) is based off a combinatorial game that is finite in length (the players will eventually run out of moves if enough moves are made in the game). At any given point in time, Left and Right have balances $1, r$, and the game is at a certain position. On each turn, Left and Right simultaneously bid. Left bids a nonnegative number $a < 1$, and similarly Right bids a nonnegative number $b < r$. If $a > b$, then Right must make a move in the combinatorial game, and a is subtracted from Left's balance. If $a < b$, then Left must make a move in the combinatorial game, and b is subtracted from Right's balance. If $a = b$, then nothing is done and bidding resets. It is possible for a game to last arbitrarily long if both players continue bidding equal amounts. If, at any point in the game, both players tie their bet, the game will immediately be declared as a draw. Otherwise, the player who cannot move loses.

2 The Draw Interval

We start with a fundamental result.

Theorem 2.1. *Suppose Left's balance is a and Right's balance is b, and the game position is G. Let $r = \frac{b}{a}$. Then, there exists positive c, d such that Left wins if $r < c$, draws if $r \in [c, d]$, and loses if $r > d$. We call $f(G) = [c, d]$ the draw interval (or dividing point if $c = d$).*

Proof. First, we show that if Right can win a game G with a balance ratio r, then Right can also win a game G with a balance ratio $s > r$. This is rather obvious, as if Right has a winning strategy (bets and moves) with balance ratio r, Right can copy this exact strategy if the balance ratio is s. Since $s > r$, Right will always have a greater balance ratio than if they had balance ratio r, and thus Right will eventually win.
Next, we show that there exists a x such that if $r \leq x$ then Right will lose. We

can set $x = 0$ for this purpose, in which case Right will be forced to make moves until they run out of them and lose.

Since the set of all r such that Right wins with balance r is bounded below, by the Greatest Lower Bound Property, it must also have a greatest lower bound, which we set d to be equal to.

We also show that if Right loses a game with balance ratio r, then Right also loses the game with any balance ratio $t < r$. This is also rather obvious. Suppose Right could win with balance t. Then, since $r > t$, we know that Right will also win with balance r, but this is a contradiction.

We also show that there exists a y such that if $r \geq y$ then Right will win. If we set y to be an arbitrarily high real number, Right can continue betting 2 and forcing Left to make moves until they lose.

Since the set of all r such that Right loses with balance r is bounded above, by the Least Upper Bound Property, it must also have a least upper bound, which we set c to be equal to.

Now suppose $r \in [c, d]$. Since $r \geq c$, Left cannot win. Since $r \leq d$ as well, Right also cannot win. Therefore, the game must end in a draw. □

In this paper, we aim to find a method to compute such c, d for a given game. For convenience, in the rest of the paper we will assume that Left has a starting balance of 1.

2.1 Numbers

We start with the simplest games, the numbers. In the numbers, $c = d$. First off, we define the simplest game, $0 = (\ |\ )$. The draw point of 0 is easy to calculate:

Theorem 2.2. $f(0) = 1$

Proof. Both Left and Right would bet their entire balances, as winning the bet is an immediate win. This means that whoever has a bigger balance will win, or, if their balances are equal, the game will end in a draw. Therefore, we have $f(0) = 1$ □

We now define two more games, $1 = (0 \mid\ )$ and $-1 = (\ \mid 0)$. We can also calculate their draw points:

Theorem 2.3. $f(1) = 2$ *(which, by symmetry, implies* $f(-1) = \frac{1}{2}$*).*

Proof. Without loss of generality let Left's balance be 1 and Right's balance be r. This works as we can divide by Lefts balance, in case it is not one. Right cannot bet an amount greater than $r-1$, as then winning would send Right's balance below 1, and Right would lose after Left moves to 0. On the other hand, Right cannot bet an amount less than 1, as if Left bets 1, Right will lose. Thus, if $r-1<1$, or $r<2$, Right will lose. If $r=2$, then Right must bet exactly one, and the game will end in a draw. If $r>2$, then Right can bet b such that $1<b<r-1$, and Right will win, so we have $f(1)=2$ □

In general, we inductively define $n=(n-1 \mid 0)$ and $-n=(0 \mid -n+1)$, where n is a positive integer. We can get an explicit formula for the draw point of n.

Theorem 2.4. *$f(n)=n+1$ if n is a positive integer (again, by symmetry, this implies $f(-n)=\frac{1}{n+1}$).*

Proof. Suppose that Right's balance is r. We proceed with induction. We already have base case of $f(1)=2$. We show that given $f(n-1)=n$, $f(n)=n+1$.

We know that Left has balance 1 and Right has balance r. Right must bet at least 1, as otherwise Right will lose immediately if Left bets 1. However, if Right bets more than 1, their balance after winning and forcing Left to move to $n-1$ is at most $r-1$. By our inductive hypothesis, $f(n-1)=n$, so Right wins this if $r-1>n$ and loses if $r-1<n$. Thus, Right wins n if $r>n+1$ and loses if $r<n+1$. In the case that $r=n+1$, Right needs to bet exactly 1, which will lead to a draw. Thus we are done. □

In fact, we can generalize this approach:

Theorem 2.5. *Let $G=(G^L \mid G^R)$. Suppose that $f(G^L)=a$ and $f(G^R)=b$, with $a<b$. Then $f(G)=\frac{b(a+1)}{b+1}$.*

Proof. Right cannot bet more than $r-a$, as then if Right wins the bet Right will be left with less than a and lose after Left moves to G^L. Right also cannot bet less than $1-\frac{r}{b}$, as if Left bets $1-\frac{r}{b}-\epsilon$ and beats Right's bet, Left will force Right to move to G^R and $b(1-(1-\frac{r}{b}-\epsilon))>r$, so Right will lose. If $r-a<1-\frac{r}{b}$, or $r<\frac{b(a+1)}{b+1}$, Right will lose. On the other hand, if $r-a>1-\frac{r}{b}$, so $r>\frac{b(a+1)}{b+1}$ it follows that Right can bet some $1-\frac{r}{b}<x<r-a$ and win. If $r-a=1-\frac{r}{b}$, then Right cannot guarantee victory, but can guarantee a draw by betting $r-a=1-\frac{r}{b}$.

We have two special cases, one where G^L is empty and another where G^R is empty game. If G^L is empty, by convention, we set $a=0$, so now $f(G)=\frac{b}{b+1}$ becomes the dividing point. If G^R is empty, then no such b exists, but we can determine that $a+1$ is the dividing point. And if both G^L and G^R are empty, then $G=0$ and we have already handled it.

□

Note that classically, $(0|2) = (0|)$ but this is not the case once we add bidding, as $f(0 \mid \,) = 2 \neq \frac{3}{2} = f(0 \mid 2)$
So far, we have only seen draw points, but no draw intervals. Draw intervals will come into play once we talk about switches.

2.2 Switches

In the numbers, it is never advantageous to make a move. However, this need not be the case. In a switch, it may be advantageous to lose a bet, which leads to more interesting play and is illustrated in the following result:

Theorem 2.6. *Let $G = (G^L|G^R)$. Suppose that $f(G^L) = a$ and $f(G^R) = b$, where $a \geq b$. Then $f(G) = [b, a]$. In particular, if $f(G^L) = f(G^R)$, then $f(G) = f(G^L) = f(G^R)$.*

Proof. Suppose that Right has balance r. Both Left and Right want to lose the bet by betting 0, but if they both do this, the game will end in a draw. If $r > a$, then Right can bet $\epsilon < r - a$, as Right will still win even if they win the bet. Similarly, if $r < b$, Left can bet $\epsilon < 1 - \frac{r}{b}$, as Left will also win even if they win the bet. However, if $r \in [b, a]$, then both players cannot risk losing the bet, so they will both bet 0 and draw. Therefore, $f(G) = [b, a]$. □