THE SURREAL NUMBERS AND OMNIFIC INTEGERS

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In this paper I define a certain type of sum of surreal numbers indexed by ordinals. I prove that each surreal number has a unique Conway normal form, a representation as the type of sum mentioned above. I give a method for calculating the Conway normal form of a surreal number given the normal forms of its options. I define the omnific integers and prove a way of determining whether a surreal number is an integer given its Conway normal form. I prove some results on omnific integers. I will assume basic familiarity with the surreal numbers.

1. Surreal Numbers

Theorem 1.1. *We can express each surreal number x uniquely in the form*

$$\sum_{\beta<\alpha} \omega^{y_\beta} \cdot r_\beta$$

where α is some ordinal, the numbers r_β are nonzero real numbers, and the y_β's are a decreasing sequence of surreal numbers (i.e. if $\beta < \beta_0 < \alpha$, then $y_\beta > y_{\beta_0}$). A sum of this form is called the Conway Normal Form of x.

To make this theorem clear, we need to define the infinite sum it includes.

Definition 1.2.

$$\sum_{\beta<\alpha} \omega^{y_\beta} \cdot r_\beta = \left\{ (\sum_{\beta<\alpha'} \omega^{y_\beta} \cdot r_\beta) + (\omega^{y_{\alpha'}} \cdot r_{\alpha'})^L \;\middle|\; (\sum_{\beta<\alpha'} \omega^{y_\beta} \cdot r_\beta) + (\omega^{y_{\alpha'}} \cdot r_{\alpha'})^R \right\}$$

for all ordinals $\alpha' < \alpha$. In other words, this sum is really an ordinal sum: the options are to move in a term of the sum and delete all terms to the right. Note that the terms must be in their usual form (as defined by multiplication and the omega map): from $\omega - 2$, Left may not move to $\omega - 1$ even though $\omega = \{\omega - 1 \mid \omega + 1\}$. The valid moves are to reals r. However, the result does not depend on the form of the coefficients or exponents.

This definition works because the ordinal involved decreases with each step. It is still required to check that the result is actually a number.

Lemma 1.3. *ω^a is infinite for all positive a.*

Proof. Since a is positive, it must have a nonnegative left option. If a has a left option 0, ω^a has left options $r\omega^0$ for all reals r, so ω^a is infinite. Otherwise, a has a positive left option, so ω^a has a left option ω^{a^L} which, inductively, is infinite, so ω^a itself is infinite. ∎

Lemma 1.4. *If $a < b$, ω^a is infinitesimal with respect to ω^b.*

Proof. For all reals r, $\omega^b - r\omega^a = \omega^{b-a}(\omega^a - r)$, which is infinite by the previous lemma. ∎

Lemma 1.5. *When in the usual form, $(\omega^a \cdot r)^R - \omega^a \cdot r$ is on the order of a real multiple of ω^a, and so is $\omega^a \cdot r - (\omega^a \cdot r)^L$ (for certain options which dominate the ones with larger differences). In fact, when x is real and y is an omega power with a left option, $xy = \left\{x^L y + xy^L - x^L y^L \,\middle|\, x^R y + xy^L - x^R y^L\right\}$.*

Proof. This is easily seen by using the definition of multiplication and omega map. ■

Now we are ready to show that the infinite sum defined above is a number. By induction on α, all of the options are numbers; thus it remains to show all the left options are less than all the right options. Consider the left option $(\sum_{\beta<\alpha_0} \omega^{y_\beta} \cdot r_\beta) + (\omega^{y_{\alpha_0}} \cdot r_{\alpha_0})^L$ and right option $(\sum_{\beta<\alpha_1} \omega^{y_\beta} \cdot r_\beta) + (\omega^{y_{\alpha_1}} \cdot r_{\alpha_1})^R$. WLOG let $\alpha_1 > \alpha_0$ ($\alpha_1 < \alpha_0$ is similar, and $\alpha_1 = \alpha_0$ is trivial). Then $(\sum_{\beta<\alpha_1} \omega^{y_\beta} \cdot r_\beta)$ has as a right option, and so is greater than, $(\sum_{\beta<\alpha_0} \omega^{y_\beta} \cdot r_\beta) + (\omega^{y_{\alpha_0}} \cdot r_{\alpha_0})^R$. The difference between this and the original left option is $(\omega^{y_{\alpha_0}} \cdot r_{\alpha_0})^R - (\omega^{y_{\alpha_0}} \cdot r_{\alpha_0})^L$, which is positive and on the order of $\omega^{y_{\alpha_0}}$, making it infinite relative to $\omega^{y_{\alpha_1}}$. Thus $(\sum_{\beta<\alpha_1} \omega^{y_\beta} \cdot r_\beta)$ is greater than $(\sum_{\beta<\alpha_0} \omega^{y_\beta} \cdot r_\beta) + (\omega^{y_{\alpha_0}} \cdot r_{\alpha_0})^L$ by an amount infinitely greater than $-(\omega^{y_{\alpha_1}} \cdot r_{\alpha_1}) > -(\omega^{y_{\alpha_1}} \cdot r_{\alpha_1})^R$, so $(\sum_{\beta<\alpha_1} \omega^{y_\beta} \cdot r_\beta) + (\omega^{y_{\alpha_1}} \cdot r_{\alpha_1})^R > (\sum_{\beta<\alpha_0} \omega^{y_\beta} \cdot r_\beta) + (\omega^{y_{\alpha_0}} \cdot r_{\alpha_0})^L$, and thus all right options are greater than all left options, and the infinite sum is a number.

It can be shown fairly easily that $\sum_{\beta<\alpha+1} \omega^{y_\beta} \cdot r_\beta = (\sum_{\beta<\alpha} \omega^{y_\beta} \cdot r_\beta) + \omega^{y_\alpha} \cdot r_\alpha$, and more generally, $\sum_{a<\alpha+\beta} \omega^{y_a} \cdot r_a = (\sum_{a<\alpha} \omega^{y_a} \cdot r_a) + (\sum_{a<\beta} \omega^{y(a+\alpha)} \cdot r_{a+\alpha})$, allowing sums to be split apart.

Lemma 1.6. *If $A < B$, $A \leq \left\{A^L \,\middle|\, B^R\right\} \leq B$ (A^L, B^R possibly representing multiple or zero options).*

Proof. $A \leq \left\{A^L \,\middle|\, B^R\right\}$ means Left wins going second in $\left\{A^L \,\middle|\, B^R\right\} - A$. If right moves in the $-A$ to $-A^L$ leaving $\left\{A^L \,\middle|\, B^R\right\} - A^L$, Left responds by moving in the other component to A^L and leaving 0. If Right moves to some B^R he leaves a positive value since $B^R > B > A$. Thus Left wins going second. $\left\{A^L \,\middle|\, B^R\right\} \leq B$ is similar. ■

Now we are ready to prove that each surreal number has a unique Conway normal form.

Proof. For a game $A = \sum_{\beta<\alpha} r_\beta \omega^{y_\beta}$, let $A_\gamma = r_\gamma \omega^{y_\gamma}$. Let $G = \left\{G^L \,\middle|\, G^R\right\}$ be a game. Inductively, let each of the options of G be written in Conway normal form. We define a new normal form K in several steps: defining the common terms, the exceptional term, and the remaining terms.

Step 1: for each β such that there are left options G^L and G^R with $G^L_\gamma = G^R_\gamma$ for all $\gamma \leq \beta$, $K_\beta = G^L_\beta$.

Step 2: Let α be the least ordinal a for which K_a is not defined. Certain options G' may have $G'_\beta = k_\beta$ for all $\beta < \alpha$. If none of these options exist, we are done. If all these options are left options or all are right options, skip to step 3. Otherwise, both left and right options of this type exist. Eliminate all options not of this type (they are dominated). Consider the αth term of these options (adding a 0 term if necessary). If it is negative for all left options and all positive for right options, then we are done. Suppose this is nonnegative (nonpositive) for both left and right options. Then define $y_\alpha = \left\{a, a^L \,\middle|\, b, b^R\right\}$ ($\left\{b^L, b \,\middle|\, a, a^R\right\}$) for exponents a of left options and b of right options, where a is included if $\{r|\} = \omega$ ($\{|r\} = -\omega$) for coefficients r of the a term, a^L otherwise, b included if $\{0|r\} = \frac{1}{\omega}$ ($\{r|0\} = -\frac{1}{\omega}$). If this does not equal any a or b, let $r_a = 1$ (-1) and we are done. Otherwise, let S be the set of all options of G with exponent y_α of the α term. Let S' be the set of left options in S with positive $\alpha+1$st coefficients and $\left\{e, e^L \;\middle|\; y_\alpha^R\right\} = y_\alpha$ for $\alpha+1$st exponent e, e included instead of e^L if $\{r\mid\} = \omega$ or right options with negative next coefficient satisfying the same condition, e included if $\{\mid r\} = -\omega$. Let $r_\alpha = \{a^L, b|c, d^R\}$ ($\{d^R, c|b, a^L\}$) where a, b, c, d are coefficients in αth terms of elements of $S \cap G^L$, $S' \cap G^L$, $S' \cap G^R$, $S \cap G^R$, respectively, unless this is not a real number. This is only possible if some b (c) is the infimum of the right options or some c (b) is the supremum of the left options (it cannot be both or the entire αth term would coincide, contradicting the definition of α), in which case let r_α be that value. If r_α equals some a or d continue to step 3; otherwise we are done.

Step 3: Let k be the exponent of the exceptional term created in step 2. Let α be the least ordinal for which the corresponding term of K is not defined. There right or left options of G whose terms coincide with K for all smaller ordinals, but not both. Suppose there are left (right) options. Consider their αth terms. If all of these are negative (positive), we are done. If there is a left (right) option without such a term and all others have that term negative (positive), then define $y_\alpha = \{\mid k\}$, $r_\alpha = 1$ (-1), and we are done. Otherwise, eliminate all options for which that term is negative (positive), and let $y_\alpha = \left\{e, e^L \;\middle|\; k\right\}$ for exponents e of these terms, where e is included if $\{r|\}(\{|r\}) = \omega$ $(-\omega)$ for coefficients r of the ω^e terms. If this does not equal any e, let $r_\alpha = 1$ (-1) and we are done. Otherwise, let let S be the set of all left (right) options of G with exponent y_α of the α term. Let S' be the set of options in S with positive (negative) $\alpha+1$st coefficients and $\left\{e, e^L \;\middle|\; y_\alpha^R = k\right\} = y_\alpha$ for $\alpha+1$st exponent e, e included instead of e^L if $\{r|\}(\{|r\}) = \omega$ $(-\omega)$ for coefficients r of the ω^e terms. Let $r_\alpha = \left\{a^L, b \;\middle|\; \right\}$ $(\left\{ \;\middle|\; a^R, b\right\})$ where a, b are coefficients in the αth terms of elements of S, S'. If this does not equal any b, we are done. Otherwise, define K_β for ordinals $\beta > \alpha$ using step 3.

The process must terminate at some ordinal because the options of G do; thus we have defined K. Note that since $\left\{e, e^L \;\middle|\; y_\alpha^R\right\} = y_\alpha$ for options not eliminated at the $\alpha+1$st step forces the birthdays of the exponents to decrease, step 3 only needs to be repeated finitely many times before the process terminates.

This formula seems more complicated than it is; essentially, the formula is to make the next term 0 if possible; otherwise, choose a simpler exponent between that of the (coinciding) left and right options if possible; otherwise, choose the simplest of the two; choose the simplest possible coefficient if this allows the next exponent to be simpler than this one; otherwise, choose a different coefficient not equal to that of the options, if possible; otherwise, continue the process. I will give an example to clarify.

To evaluate $\left\{\omega^3 + \omega^2 + \omega + \omega^{\frac{1}{2}} \;\middle|\; \omega^3 + (1 + \frac{1}{2^n})\omega^2 + \omega\right\}$, ω^3 is a common term, so $y_0 = 3$, $r_0 = 1$. There are no left and right options with the same first and second terms, so we move to step 2. The coefficients are nonnegative for both types of options, so we define $y_1 = \{1\mid\} = 2$. This does equal exponents of both left and right options. All options are in S. $\left\{e^L \;\middle|\; y_1\right\} = \{0 \mid 2\} = 1 \neq 2$, so no options are in S'. Thus $r_1 = \left\{0 \;\middle|\; 1 + \frac{1}{2^n}\right\} = 1$. This makes the exceptional term coincide with that of a left option, so we move to step 3. We have a positive next term, ω, so we let $y_2 = \{0 \mid 2\} = 1$. This equals the exponent of our term, so the left option is in S. It is also in S', since $\left\{(\frac{1}{2})^L \;\middle|\; 2\right\} = \{0 \mid 2\} = 1$. Thus $r_\alpha = \{1\mid\} = 2$. This does not equal 1, the coefficient of the option, so we are done, and the result is $\omega^3 + \omega^2 + 2\omega$. That this indeed equals $\left\{\omega^3 + \omega^2 + \omega + \omega^{\frac{1}{2}} \;\middle|\; \omega^3 + (1 + \frac{1}{2^n})\omega^2 + \omega\right\}$ can be checked by playing the difference.

Now I will show $G = K$. The proof amounts to checking the various cases of the formula. I will show that $G^L < K < G^R$ for all options G^L, G^R of G, and for all options K^L, K^R of K, there exist options of G with $K^L \leq G^L$, $K^R \geq G^R$, which implies $G = K$. For the former, the terms in the normal forms of the options of G coincide with those of K until K has a term that is greater (less) for left (right) options.

For the latter, first consider the right options. They are of the form $K^R = (\sum_{\beta<\alpha} r_\beta \omega^{y_\beta}) + (r_\alpha \omega^{y_\alpha})^R = A + (r_\alpha \omega^{y_\alpha})^R$. If there is some G^R coinciding with K for the first α terms, then $K^R > G^R$ because the αth term is greater and all previous terms are the same.

Now consider the case where no such option exists. This eliminates moving in one of the coincident terms. The right option may be moving in the exceptional term or one of the terms from step 3.

Suppose the option is moving in the exceptional term. Let A be the sum of corresponding terms. Then it may be of the form $A + (\omega^{\left\{a, a^L \middle| b, b^R\right\}})^R$ which equals $A + r\omega^b$, allowing $G^R = A + \frac{r}{2}\omega^b +$ [smaller terms], or $A + r\omega^{b^R}$ allowing $G^R = A + \omega^b +$ [smaller terms]. It may also be of the form $A + (\{a^L, b | c, d^R\}\omega^{y_\alpha})^R +$ [smaller terms] which can equal $A + d^R\omega^{y_\alpha} +$ [smaller terms], allowing $G^R = A + d\omega^{y_\alpha} +$ [smaller terms], or $A + c\omega^{y_\alpha} + (r_\alpha - c)\omega^{y_\alpha^L} +$ [smaller terms], allowing $G^R = A + c\omega^{y_\alpha} + x\omega^{y_\alpha^L} +$[smaller terms] for sufficiently small x. Finally, it may be of the form $A + (b\omega^{y_\alpha^R})^R = A + b^R\omega^{y_\alpha} +$ [smaller terms], with b the infimum of the cs, d^Rs, allowing $G^R = A + c\omega^{y_\alpha} +$ [smaller terms].

Suppose some right options coincided with K in initial terms for step 3. Then $r_\alpha\omega^{y_\alpha}$ must be the final term of K, so we consider the terminating conditions for step 3. K^R can be $A + (-\omega^{\left\{ \middle| k\right\}})^R = A$, allowing $G^R = K \leq K$. It can be $A + (-\omega^{\left\{e, e^L \middle| k\right\}})^R$ which equals $A - r\omega^e$, allowing $G^R = A - (r+1)\omega^e$, or $A - r\omega^{e^L}$, allowing $G^R = A - k\omega^e +$ [smaller terms]. It can be $A + (\left\{ \middle| a^R, b\right\}\omega^{\left\{e, e^L \middle| k\right\}})^R = A + (\left\{ \middle| a^R, b\right\}\omega^{y_\alpha})^R$ which equals $A + a^R\omega^{y_\alpha} + (r\left\{ \middle| a^R, b\right\} - a^R)\omega^{y_\alpha^L}$, allowing $G^R = A + a\omega^{y_\alpha} +$ [smaller terms], or $A + b\omega^{y_\alpha} + (r\left\{ \middle| a^R, b\right\} - b)\omega^{y_\alpha^L}$ allowing $G^R = A + b\omega^{y_\alpha} + k\omega^e +$ [smaller terms] for sufficiently small k (the options of y_α are e, allowing arbitrarily small k, or e^L). Thus we have finished this case.

Now suppose some left options coincided with K in initial terms for step 3. Then there exist options G^R coinciding with K until the exceptional term of K, which is less than the coinciding term of G^R. Then the options K^R are $A + x\omega^k + B + (r\omega^{\left\{x \middle| k\right\}})^R = A + (x + r - r^L)\omega^k + B + r^L\omega^{\left\{x \middle| k\right\}}$ for positive reals s, where A represents coinciding terms, $x\omega^k$ the exceptional term, and B further terms. This allows $G^R = A + x\omega^k +$ [smaller terms].

The proof is the same for left options.

Thus every surreal number can be represented in Conway normal form. To prove that this representation is unique, if any coefficient or exponent is increased (decreased), the difference in that term is infinite with respect to all following terms, so the entire sum in increased (decreased). ■

Conway defined this type of infinite sum differently. His definition is as follows: given a surreal number x, let $r_0\omega^{y_0}$ be the unique real multiple of a power of omega commensurate with x (not infinitesimal or infinite with respect to it), and say it is the 0 term of x. Then suppose for some α we have defined the β term of x for all $\beta < \alpha$. We declare $\sum_{\beta<\alpha} \omega^{y_\beta} \cdot r_\beta$ to be the simplest (least birthday) number with all the same β terms as x. We then write

$x = \sum_{\beta<\alpha} \omega^{y_\beta} \cdot r_\beta + x_\alpha$, terminating if x_α is zero, and otherwise defining the alpha term of x to be the unique $r_\alpha \omega^\alpha$ commensurate with x_α. It is not hard to show that the sum defined in this paper is equivalent; a sum of the type considered in this paper is commensurate to its first term, providing a base case; when α is a successor ordinal, $\sum_{\beta<\alpha} \omega^{y_\beta} \cdot r_\beta = (\sum_{\beta<\alpha-1} \omega^{y_\beta} \cdot r_\beta) + \omega^{y_{\alpha-1}} \cdot r_{\alpha-1}$ for both types of sums; and when α is a limit ordinal, the surreal numbers between the options of $\sum_{\beta<\alpha} \omega^{y_\beta} \cdot r_\beta$ are exactly those with β term $\omega^{y_\beta} \cdot r_\beta$, so the simplest among them, i.e. the Conway sum, must equal the ordinal sum by the simplicity theorem.

2. Omnific Integers

Definition 2.1. The omnific integers are the surreal numbers x with $x = \{x - 1 \mid x + 1\}$.

This formalizes the intuition that any number should be within 1 of an integer.

Theorem 2.2. *A surreal number is an omnific integer iff all exponents in its Conway normal form are nonnegative and the ω^0 term is a normal integer.*

Proof. Write $x - 1$ and $x + 1$ in Conway normal form. Using the method I proved above for determining the normal form of the result, all terms in the normal form of x with positive exponent will be the same for $\{x - 1 \mid x + 1\}$. The exceptional term is the ω^0 term. Let r be the constant term of x; then the exceptional term will be $\{r - 1 \mid r + 1\}$. There will be no further terms. Thus $x = \{x - 1 \mid x + 1\}$ iff there are no terms in the normal form of x with negative exponent, and the constant term r of x satisfies $r = \{r - 1 \mid r + 1\}$. By the simplicity theorem, this happens exactly when r is an integer; thus we have proven the theorem. ■

Each irreducible regular integer is also an irreducible omnific integer; any factorization of an integer cannot have any omega powers, leaving only integer factorizations possible; thus irreducible integers remain irreducible omnific integers.

Theorem 2.3. *For all omnific integers n with a finite number of terms in their Conway normal form that are not regular integers, if there is a real r such that all exponents in the Conway normal form of n are r times a rational, n is reducible.*

Proof. Let n be $\sum_i r_i \omega^{r\frac{p_i}{q_i}}$. Let $m = \text{lcm}(q_i)$. Make the substitution $t = \omega^{\frac{r}{km}}$. Then $n = \sum_i r_i x^{\frac{kp_i m}{q_i}}$, a polynomial in x with coefficients in $\mathbb{R}$ with degree as large as desired. By the fundamental theorem of algebra, this polynomial has a nonconstant factor, thus giving infinitely many factorizations as k is increased. ■

For example, $\omega+1$ can be factored as $(\omega^{\frac{1}{3}}+1)(\omega^{\frac{2}{3}}-\omega^{\frac{1}{3}}+1)$, $(\omega^{\frac{1}{5}}+1)(\omega^{\frac{4}{5}}-\omega^{\frac{3}{5}}+\omega^{\frac{2}{5}}-\omega^{\frac{1}{5}}+1)$, etc.

Theorem 2.4. *Every surreal number is a quotient of omnific integers.*

Proof. For a surreal x, let the exponents in the terms of x be y_α. Let $k = \{\mid y_\alpha\}$. k is less than all y_α, so $\frac{x}{\omega^k}$ only has positive terms, and thus is an omnific integer. $k \le 0$, so ω^{-k} is an omnific integer, so $x = \frac{\frac{x}{\omega^k}}{\omega^{-k}}$. ■

The Conway normal form for surreal numbers allows them to be viewed as generalized power series, or Hahn series: $No = \mathbb{R}((No))$. The omnific integers can then be represented as $\mathbb{Z} \bigoplus \mathbb{R}((No^{<0}))$. We can apply the following theorem from generalized power series, from [1]:

Theorem 2.5. *Let K be a field of characteristic 0, G a divisible ordered abelian archimedean group. Suppose that $a \in K((G^{\leq 0}))$ (the ring of series with coefficients in K and negative exponents in G) is not divisible by any monomial t^γ with $t < 0$. If the order type of the support (the set of exponents) of G is either ω or of the form ω^{ω^β}, then both a and $a+1$ are irreducible.*

The exponents are nonnegative, unlike what we are used to, because of a change of variables $t = \omega^{-1}$. This leads to elements of the ring having well-ordered support.

This result does not immediately apply to the omnific integers, since the surreals are not archimedean. However, we can use it to prove the following.

Theorem 2.6. *$\omega + \omega^{\frac{1}{2}} + \omega^{\frac{1}{3}} + ... + 1$ is irreducible over the omnific integers. In general, if an omnific integer a has real exponents and is not divisible by any monomial ω^r with real r, and its sum has limit ω^{ω^β}, then $a+1$ is irreducible.*

Proof. We can expand the ring we are considering by adding elements with arbitrary real constant terms. Any factorization in the omnific integers immediately implies factorization in this larger ring, and any factors in the larger ring can be divided by reals to get omnific integers.

Since terms of omnific integers cannot have negative exponents, if there are any infinite exponents in x or y, there are infinite exponents in xy. Thus we only need to consider omnific integers with exponents less than ω, i.e. factorization in the ring $\mathbb{R}((No^{-\omega < x \leq 0}))$. Elements of $No^{-\omega < x \leq 0}$ can be written uniquely as a real part and infinitesimal part, and these representations have pairwise addition, thus (under addition) $No^{-\omega < x \leq 0}$ is isomorphic to $\mathbb{R} \oplus K$, where K is the ring of infinitesimal surreal numbers. An exponent $r\omega^{s+k}$ with reals r, s, infinitesimal k can be written as $r\omega^k\omega^s$, and like terms can be combined based on the ω^s term, so $No^{-\omega < x \leq 0} \cong \mathbb{R}^{<0} \oplus K \cup K^{\leq 0}$, $\mathbb{R}((No^{-\omega < x \leq 0})) \cong \mathbb{R}((\mathbb{R}^{<0} \oplus K \cup K^{\leq 0}))$. Now let us consider the larger ring where positive elements of K are thrown in, i.e. $\mathbb{R}((\mathbb{R}^{\leq 0}))((K))$. In this ring, our series is not divisible by a monomial, its order type is still the same, and $\mathbb{R}$ satisfies the criteria for G, thus by Theorem 2.5 $a+1$ is irreducible in this ring. This does not show irreducibility in the original ring, because the units are different: it is possible $a+1$ could factor as bc, $b \in \mathbb{R}((K))$, $b \notin \mathbb{R}$, $b, c \in \mathbb{R}((\mathbb{R}^{<0} \oplus K \cup K^{\leq 0}))$, thus $b \in \mathbb{R}((K^{\leq 0}))$. Note $\mathbb{R}((K))$ is a field. Suppose for contradiction that it does. Then $(a+1)b^{-1} = ab^{-1} + b^{-1} = c$. Since $b \notin \mathbb{R}$, b has some negative exponent, thus b^{-1} must have some positive exponent. All exponents in ab^{-1} are less (more absolute value) than any infinitesimal, thus any exponents in b^{-1} are present in c, so c has a positive exponent, which is a contradiction. Thus $a+1$ is irreducible in the smaller ring and so in the omnific integers. ■

3. Sources

1. Berarducci, A.. "Factorization in generalized power series." Transactions of the American Mathematical Society 352 (1999): 553-577.
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