THE ANGEL AND DEVIL PROBLEM

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Abstract. This article will discuss the **The Angel and Devil Problem** in the subject of combinatorial game theory. We will go over the game, the problem, and then discuss a winning strategy and strategies that appear to win, but don't.

1. Introduction

We will begin by giving an overview of the **Angel and Devil Game**, and then going over the **Angel and Devil Problem**. The **Angel and Devil Game** is a combinatorial game involving 2 players, one of them being the Angel, and one of them being the Devil, on an infinite chessboard. The Angel will move first, and will move for some k from (x, y) to any point (x', y') such that $(x', y') \neq (x, y)$ and $|x' - x| \leq k$, $|y' - y| \leq k$. k is called the power of the angel, and an angel with a power of k will be called a k-angel. The Devil will move second, and each turn will take a square on the board and remove it, so that an angel can no longer move to that square, but can move over that square because the Angel is flying. The only restriction on which square is removed is that it can't be the square the angel is currently standing on. In order to win as the Angel, one must be able to move indefinitely, and never be trapped by the Devil. In order to win as the Devil, one must trap the k-angel in a square of width $2k + 1$. The **Angel and Devil Problem** reads, is there a k such that a k-angel can win the game? In this article, we will discuss this problem and the general problem of which k's will win for which players.

2. Basic Strategies

We will begin with the following basic, but important observation.

Proposition 2.1. *If a k-angel can win, then a k'-angel can also win, for any $k' \geq k$.*

Proof. We will give an informal proof to this observation. First, if the angel will not get trapped in a square of width $2k+1$, then it will not get trapped in a square of greater width because the amount of extra squares needed to be removed gives the Angel enough time to escape the square, especially considering the increased maneuverability of the angel. ■

Now we will move onto basic strategies that seem to work, but do not actually.

Algorithm 2.2. We will create a potential function which will change depending on squares already removed. Every move will greedily go to a square which minimizes the potential function.

The reason this general algorithm does not work is because the Devil will always find a way to exploit the strategy by finding features of the function which make it sensitive to certain removed blocks. We will show this with an example.

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Example. The potential function will increase most by squares near the angel, but not so much by squares farther away from the angel.

This example is of a potential function sensitive to squares near it. So the devil can exploit the sensitive squares with a trap. The trap will be a large amount of removed squares in the shape of horseshoe with a big opening, and will be extremely far away from the angel. At any time that we want, we can locate the angel, and remove a square extremely close to it. In order to minimize the potential function, the angel will move in the exact opposite direction of the square we just removed. So we can take advantage of this and lure the angel into the horseshoe trap, and then close off the horseshoe. We can always change the size of the horseshoe to make this work. So this is one example of taking advantage of the function's sensitive squares.

Example. The potential function will increase most by squares very far away from it, but not so much by squares near it.

For this example, we can take advantage of squares far away from the angel, and create a horseshoe trap near the angel. We can follow the same strategy as last time by putting squares far away from the angel and luring it into the horseshoe trap, which doesn't effect the potential function that much since it is extremely close to the angel.

In general, this greedy strategy will not work becuase we can create such traps and lure the angel into them by exploiting the function's sensitive squares. The way the trap is set and shaped, and the general strategy depends on the function however.

The third greedy strategy is slightly more complicated and deals with a certain class of angels known as *fools*.

Definition 2.3. A *Fool* is an angel, who when moving from (x, y) to (x', y'), will always have the condition of $y' > y$.

Simply speaking, the fool is an angel who's y-coordinate will strictly increase with each turn. The fool's strategy is to keep going upwards, and occasionally go east or west to avoid any obvious traps. We will now see why the fool's strategy doesn't work.

Theorem 2.4. *The devil will always win if the angel uses the fool's strategy.*

Proof. First, we can see that the positions of a fool throughout the game will always be bounded by an upward cone with its vertex at the starting position of the fool. Now, we will describe the devil's strategy, and it will be drawn in (Figure 1). We will draw a line AB, at a very far vertical distance from the fool which will satisfy properties later needed in the proof, and this distance will be H. Next, the devil will remove one out of every M squares on AB. M is chosen so that the devil will get through removing squares on AB slightly before the fool reaches the distance $\frac{1}{2}H$ below AB. Now we have a scenario similar to before with the fool in the cone QCD, with CD being half the length of AB. Now the devil will remove the second of every M squares along CD, and by the time he finishes the fool should almost be at the distance $\frac{1}{4}H$ below AB. Now the fool is in the even smaller cone REF, and we will keep going like this. When the fool reaches the distance $H' = 2^{-M}H$ below AB, the devil will have removed the line segment of squares that can be reached by the fool. Now the devil will remove one out of every M squares of the line segment below this line, which will be called $A'B'$, and this will be finished slightly before the fool is a distance $\frac{1}{2}H'$ below $A'B'$, and now the devil will remove every second out of M squares on $C'D'$ which is the

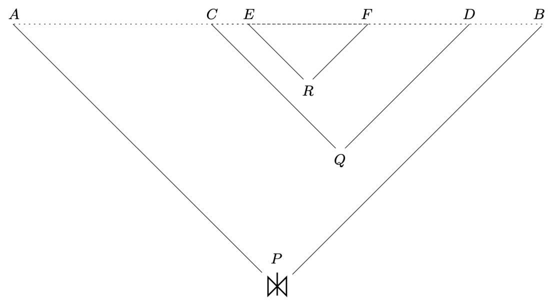

Figure 1

new subsegment of $A'B'$ that the fool can reach. He will continue the same process like this with $E'F'$ next. Now if we take H to be of the form $1000 * 2^N$ where $N > 1000M$, then by the time the fool crosses the line 1000 squares below AB, the devil will have removed all squares the fool might reach between the fool and AB, and so the fool is trapped and has no moves.

■

The final basic strategy we will cover is that of the *Lax Fool.*

Definition 2.5. The *Lax Fool* is an angel who if even is on the point (X, Y), will always move to a point (X', Y') with the restriction that $Y' \geq Y$.

To put it simply, the y-coordinate of the lax fool will not decrease, similar to the fool, or plain fool, who's y-coordinate cannot decrease or stay the same. We now have the following theorem.

Theorem 2.6. *The devil can catch the lax fool.*

Proof. First, the drawing of the devil's strategy is shown in (Figure 2). For th devil's strategy, he will use his odd-numbered moves to turn the lax fool into a regular fool. He choose two squares a suitable distance D horizontally away from the starting point of the lax fool, and calls the squares L and R. He removes squares inwardly from L and R, and alternates between the two sides. Whenever the lax fool moves upwards, the devil will do the same process but with the L and R being shifted upwards to align with the new position of the lax fool. If the lax fool stays on a horizontal line forever, we will keep removing squares forcing him to move upward. Also, there is a finite amount of moves that the lax fool can stay on a horizontal line for, and we will call it x. If the fool has power k, we can think of x moves as 1 moves, and think of the lax fool to be a regular fool with power kx, since in those x moves he can move k squares. So the devil can use his even-numbered moves to use the previous strategy to catch the plain fool with power kx.

■

This concludes analysis of basic and greedy strategies that don't work for the angel. We will now move onto the case where the angel has power 1, and moves like a chess king.

3. Angel with Power 1

Now we will discuss the case where the angel has power 1. We have the following theorem.

Theorem 3.1. *The devil can catch the* 1*-angel.*

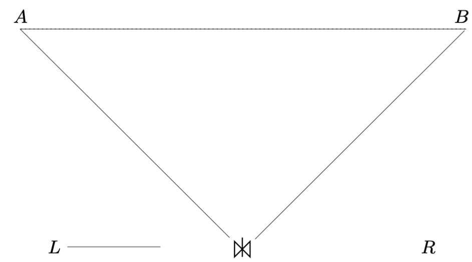

Figure 2

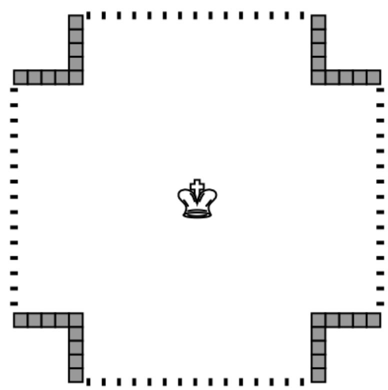

Figure 3

Proof. We will give a somewhat informal proof to this theorem. First, the devil will use his first 44 moves to remove certain squares at the corner of an imaginary box around the angel, as shown in (Figure 3). After this, we can build the border of our imaginary box to trap the angel. However, if the angel is within 5 blocks of border of our box, we will use Algorithm 3.2.

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Algorithm 3.2. First, we will discuss trapping the angel if it is right besides a segment of 3 removed border squares, with the angel being aligned with the middle border square. If the angel moves one way, the devil will remove the corresponding block on that side of the 3 block segment, and will not let the angel pass. Thus, this position is a simply win for the devil. This is important as this position will arise when using the algorithm. Next, if the angel is 5 squares away from the border of our box, then we must remove the square 5 squares away from him. Without loss of generality, assume the border is the north border of the box. We will not go into the optimal strategy for the devil for every single position the angel can go to, but we can give a general strategy to prevent the angel from passing the border. If the angel moves forward and left, remove the square on the border which is vertical one square to the left of him. If the angel moves forward and to the right, remove the square vertical one square to the right of him. If the angel moves forward, remove the square in front of him, and if that square is removed, remove one to the left or right. With this strategy, the best position the angel can have is the one described earlier, with 3 blocks in front of him, which we know is a win for the devil. This means, we can trap an angel 5 blocks from the border.

This gives an informal proof of why the devil wins when the angel has power 1. Now we will move on to actual winning strategies for the angel.

4. Winning with a 2-Angel

We will be going over is András Máthé's proof that a 2-angel can win. The 2-angel proof does not actually give a strategy and is only an existence proof. It is somewhat complicated, and so we will only give informal proofs of the most important propositions required to show the existence of a win for the 2-angel.

Definition 4.1. A *Nice Devil* is a devil who never eats a square that the angel has previously stayed on, nor a square on which the angel could've previously gone to.

Trivially, the angel will move back and forth and never stop moving.

Theorem 4.2. *If there is a strategy which entraps the p-angel into a bounded domain, then there exists a strategy to trap the p-angel into a bounded domain with a nice devil.*

We are not going to go into the proof as it has graph theory concepts beyond the scope of the article, so we will give a short summary. First, we have the following definition.

Definition 4.3. A *journey* of a p-angel is a finite sequence of squares $(u_0, u_1, u_2, u_3, \ldots, u_n)$ such that $u_0 = (0, 0)$ and $d(u_i, u_{i+1}) \leq p$ for all $0 \leq i < n$.

Now we will construct a directed graph of n nodes for the journey of our angel. For every $0 \leq i < n$, we find the least possible j such that $d(u_i, u_j) \leq p$, and construct an edge from j to i. Now there is a unique path from n to 0, because the graph has $n + 1$ nodes, and n edges which makes it a tree. This unique path will be written as $(a_k, a_{k-1}, \ldots, a_0)$. We now have the following definition.

Definition 4.4. The *reduced journey* of angel will be a tuple of the form $(v_0, v_1, v_2, \ldots, v_k)$, where for $0 \leq i \leq k$, we have $v_i = u_{a_i}$.

We will now summarize the strategy for the nice devil. Given a mapping ϕ for the devil which maps the angel's journey, u, to a sequence of squares to be removed, we can create a winning strategy for the nice devil by removing squares the regular devil would have removed in the reduced journey v. And if the nice devil cannot remove certain squares in $\phi(v)$, then he is not required to. We will not prove this strategy works for the nice devil.

From here, we will not prove how the 2-angel wins. We will only give the following outline of the proof. First, we prove that a *runner* (which we will not define) will beat a nice devil. Next, we slightly modify the runner to become a 2-angel. Next, we prove properties of the 2-angel that the runner also has, and show this is sufficient for the 2-angel to beat the nice devil. Finally, we will show that the 2-angel beating the nice devil implies that the 2-angel can beat the devil. This gives the basic outline for Máthé's 2-angel proof. [1]

References

[1] Conway, John H. *The angel problem.* Games of no chance, 29, pp. 3-12, 1996. MSRI Publications.
[2] Máthé, András. *The Angel of power 2 wins.* Combinatorics, Probability and Computing, 3, pp. 363-374. 2007.
[3] Kutz, Martin. *The Angel Problem.* Refebium, Freie Universität Berlin. 2004.