3 Player Games

Arindam Kulkarni

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1 Introduction

A logical follow up to the standard 2 player combinatorial game is to add a player - a 3 person game. However, this proves to be far more complicated then the 2 player version. In the 2 player version, each player's aim is to stop only the other from winning, and thus their moves are ONLY dependent on the other player's moves. This is how a 'winning strategy' is developed (if there is a possibility of one) and otherwise, how we define P and N positions - where the outcome is decided on who starts, assuming players play the best move each time. However, in a 3 player game, one player's moves are not solely dependent on one other player, and thus there is not necessarily a winning strategy or P, N position which we can identify.

For example, in a 2 player game, Player 1 knows what Player 2 would move after each move (note that these players always pick the optimal move, so Player 1 would understand what Player 2 would be picking). Thus, Player 1 is able to build a winning strategy as he knows every move (if there is a winning strategy). Take a 2 player nim subtraction game with one pile of 6 stones. Player 1 knows that he's winning, and he knows that he needs to return a multiple of 4 to player 2 to win.

Now take a 3 player nim subtraction game of a pile of 6 stones, such that each player can remove 1 to 3 stones. Player 1 knows that regardless of what he puts, Players 2 and 3 could divide the remaining stones between them and thus allowing Player 2 or 3 to win the match. However, he also knows that he decides the winner based on how many stones he removes - if he removes 1 or 2 stones, he allows Player 3 to win, whereas if he removes 3 stones, he allows Player 2 to win. So he cannot possibly win, but he's still deciding the result?! How should he choose whom he wants to win? This is why there are specific rules on how to handle a player that has already recognized defeat.

2 Suggestions for Rules in 3 player games

One theory for 3 player games is that as soon as a player N has successfully figured out he/she has lost the game, that player adopts a 'revenge' strategy - playing to make one player lose based on the 3rd player's choice. This method has several positives and negatives. One of the negative looks on this method is that the player must accurately know he has lost, which is very unrealistic - it is tough to judge a player has lost with potentially many turns ahead. In addition, the player may be incorrect in their assumption they have lost - they would be effectively deciding the game based off of a potentially incorrect assumption.

Finally, if they were correct in this prediction they have lost, the idea that they should be able to decide the result of the game when they are OUT of the game is very controversial - if it is impossible for them to win, should they influence the result? However, this idea comes from the tendency of emotions to get in players' way, and how this could impact a real life game in such a scenario.

Another theory for 3 player games decrees that a player is rewarded based on how close to the end they took their last turn - if they were not able to be the last ones to play a turn, they should aim for 2nd last to come 2nd. This would give motivation for the losing player to still impact the game for his/her own result, and would fit neatly in the game system. However, there still are objections towards why the losing player is playing when he cannot win. In addition, if there is a winning strategy to place somewhere, a sensible person (which these players clearly are since they play perfect moves) would not place winnings or rewards in such a game.

In essence, 3 player games have the concern that 2 players can almost always gang up on the 3rd.

3 Basic Definitions and Rules

We must define positions in a way that allows us to show which of the 3 players is 'winning' (has a winning strategy), if there is one. A way to do this is to define each game by who's about to play, who just played, and the other player. We will call the player that has just taken a turn player P, the player about to move N, and the other player O. As each turn is taken, these roles keep switching and moving around. For example, in a single turn, initial player P will become player O, player N becomes player P, and player O becomes player N. We also define positions based on who, if anyone, has a winning strategy. A game G will be called a P game if player P at the moment has a winning strategy. Similarly, G would be called an N game or O game if N or O have winning strategies, respectively. Similar to the players switching these roles, a P game would become an O game, an O game an N game, and an N game a P game, with every turn. Thus, we get the following theorems:

Definition 3.1. A game G is an N game if it has an option that is a P game.

Proof. After N moves, he/she becomes a P player - now the last player to have moved. Thus, he/she must pick an option that leaves a P position in order for them to have a winning strategy while the next player moves. One option that does so is sufficient □

Definition 3.2. A game G is a P game if and only if every option is a O game.

Proof. Similar to the reasoning for number 1, after P moves, she becomes an O player - except unlike the other case, here she is not the one to move next. Thus, for it to be a P game, there must not be a single move that allows a different player to play a non O option, as if there is they can make sure that P doesn't win. Thus, every option must be an O option for it to be a P game. □

Definition 3.3. A game G is an O game every option is an N game.

Proof. Same as reasoning above. □

This recursive method lets us define all winning positions successfully. However, there are games that do NOT have a winning player. These games will be defined as *Queer* games Q. How would we know if a game G is a Q game or not? We simply have to check its options.

Definition 3.4. If a game does not satisfy any of these rules, it is a Q game, since then we cannot categorize it as a winning position for any of the players. The result can differ despite all players playing with the best intentions of winning, and making the best moves to do so.

4 Applications in Nim

Let's apply this to some real games. Take the nim subtraction game - from potentially multiple piles of stones, each player is allowed to take 1, 2, or 3 stones out but not more. The last player to take a stone wins. We start with the base case - 0 piles of stones. This is clearly a P position as N would not be able to take away stones and continue the match. Let's take the game with 1 stone - N would have to take away that one stone and there would be no moves left - this game is equal to $\{0\}$. Since this game has an option that is a P game, by rule (1) this must be an N game.

We know that games 2 or 3 (2 or 3 stones respectively) are also N positions as N could simply take the whole pile. In option notation, the game $G = 2$ has options for N to go to 1 stone or 0 stones. This would be written as $G = 2 = \{0, 1\} = \{P, N\}$. This game has an N game option, but by rule (1) we only need one P game option for it to be an N game. The process for showing the same for a pile of 3 stones is very similar.

What about a pile of 4 stones? $G = 4$ can go to 3, 2, or 1 stones so $G = 4 = \{1, 2, 3\} = \{N, N, N\}$. By rule (3), as every option is an N game, G is an O game. A game H of 5 stones however, would be defined by $\{2, 3, 4\} = \{N, N, O\} \ni Q$ by rule (4). We can see this by how depending on what N moves, the game will either be won by one of the other two players.

Theorem 1. Any nim game of a single pile with more then 4 stones is a Q game.

Proof. We have shown that $G = 5$ is a Q game, and we have shown 4 is a O game, and 3 is a N game. Thus, by rule 4, $H = 6 = \{5, 4, 3\}$ will be a Q game as well. Similarly, $7 = \{6, 5, 4\}$ and $8 = \{7, 6, 5\}$ are Q games too. From here on, any pile of stones more then 7 will be $G = \{Q, Q, Q\}$ and will match rule 4. This completes the proof. □

So we've shown that a player cannot have a winning strategy for a pile of more then 4 stones. However, there are more possibilities in games with multiple piles of stones. For example, take the game $G =$ 1-1 (two piles with 1 stone each). With option notation, $G = \{1\} = \{N\}$ as a player can only remove a pile by taking the stone away, leaving a pile with one stone. As every option in G is an N game, we know this must be an O game.

Theorem 2. A game G consisting of n piles of size 1 stones will be an N position if $N \equiv 1 \pmod 3$, an O position if $N \equiv 2 \pmod 3$, and a P position if $N \equiv 3 \pmod 3$

Proof. We will simply show that n piles of 1 stone will have the same position as n+3 piles of 1 stone. For this problem, call (n) notation for n piles of 1 stone. If (n) belongs to position P, N, O, (n+1) would belong to position N, O, P respectively as (n+1) would only have 1 position which by rules (1) (2) (3) would match as so. Similarly, (n+2) would belong to O, P, N respectively and (n+3) to P, N, O respectively. Thus (n+3) belongs to the same game class as (n). Since $1 \ni N, 2 \ni O, 3 \ni P$, we have proved the theorem successfully. □

What about $H = 1 - 2$? H has options to go to just 2, or 1-1 so $H = \{2,\ \text{1-1}\} = \{N, O\} \ni Q$.

However, $G = 1 - 1 - 2$ is not Q, as N wants to play to a P game option which he can simply do by taking one stone from the pile of 2, leaving $1 - 1 - 1 \ni P$. Similarly, $1 - 1 - 1 - 2$ also has a P option in $1 - 1 - 1$, and is thus an N game as well.

Theorem 3. In a game G with one pile of 2, and n piles of 1, G is an N game if and only if $n \equiv 2, 3 \pmod 3$. Otherwise, G is a Q game.

Proof. If $n \equiv 2 \pmod 3$, player N would be able to take one stone from the pile of 2, leaving $n + 1 \equiv 3 \pmod 3$ piles of 1 stone each. This, by Theorem 2, is a P option and thus the game is an N game (it only needs 1 P option to be an N game). If $n \equiv 3 \pmod 3$, N would have to take the whole pile of 2 stones, and this would leave a P option which makes it an N game as well.

If $n \equiv 1 \pmod 3$, the game would have options that are N or O games, which makes G a queer game by definition 3.4. □

5 Addition

Just like with 2 player games, the addition of 3 player games can be defined by the following:

For a game G and a game H, $G + H = \{G + H', G' + H\}$ where G', H' represent the options of G, H respectively.

This recursive option notation lets us see many things about the class of a sum's position.

Theorem 4. None of the following are possible:
(1) $O + O = P$
(2) $N + P = P$
(3) $O + P = N$
(4) $P + P = O$
(5) $O + N = O$

Proof. Let's start with the 1st statement. For (1) to be true, $O + N$ ($X + Y'$ in the summation) would have to be an O game (P's option).

For $O + N = O$ to be true, we must have both of $N + N$ and $O + P = N$ (O options).

For $O + P = N$ to be true, we must have $N + P$, or $O + O = P$.

For $N + P = P$, we must have both of $P + P, O + N = O$.

For $P + P = O$, we must have $O + P = N$.

At this point, we've shown a chain between all these 5 statements, meaning that each one cannot be true without one another. Notice that with each game we show this with through its options, the game's birthday is reduced by 1. This chain will continue spiralling down till the birthday is 0, at which point none of these can be shown to be true. Thus, through this induction on birthdays, we have completed the proof. □

Similar to these 5 addition statements, there are 13 more which can be proven with this inductive method, such that all these 13 rely on each other as well as the first 5.

6 Podium Rule Strategy

As we discussed in section 2, one method to handle queer games is the 'podium strategy' - each player plays to come closest to 1st for every move. This method makes sure that results in these Q games are not arbitrary, as a player can justify which move they pick even if they can't win. However, we must make a distinction between two kinds of queer games.

Take the game we looked at earlier - a nim game with a single pile of 5 stones. We showed that player N will not be able to win this match, no matter what he picks. This is true as $G = 5 = \{4, 3, 2\} = \{O, N, N\}$. If N plays into 4, P wins, and if he plays into 3 or 2, O wins (remember the positions shift after a move).

Now take a game H of a single pile of 6 stones. This game is also a Q game, but the options are a bit different - $H = 6 = \{5, 4, 3\} = \{Q, O, N\}$. If N lays in 3, O wins, and if N plays in 4, P wins. However, if N plays in 5, it is up to player O to decide the result of the game. While in this specific game, the podium rule means O would allow P to win, depending on the game and rules there is a chance that the Q game might go in N's favor (for example in this case if the game did not involve the podium rule). Thus, N would never play in a N or O option when she knows she has a possibility of winning the game by going for the Q option.

Definition 6.1. Call a game $G \ni Q$ a *lost* game if G has no Q options. Call it a *random* game if G does have at least one Q option.

With this definition, a nim game G of 5 stones would be a *lost* game - all options of G are O or N games and N has no chance of winning the game. However, a nim game H of 6 stones is a *random* game - it has a Q option and thus it is up to specific methods of play and potentially random choices to decide the match which could land in N's favor.

Lost games would never allow N to win as no matter what option he picks, one of the other 2 players would have a winning strategy. On the other hand, a *random* game might just allow N to win. Thus, we get the following conclusion.

Theorem 5. In *random* games, N will always play in a Q option rather than a N or O option.

Proof. N does not have a winning strategy as there are no P options. Thus, the only chance N has of winning is in the random game of Q as by the podium rule or a random pick of an option, he/she might end up winning. If N picks an N or O option, N knows he/she will not win for certain. □

If there are no P options OR Q options, or in other words the game is a *lost* game, what would N do? N knows that there is absolutely no chance of winning the game, so he goes by the podium rule and tries to come 2nd - be the 2nd last player to take out stones. To do this, N would have to be behind O, and would thus want O to win rather than P. As a result, we get the following.

Theorem 6. In a *lost* game, player N picks an N option rather than an O option.

Proof. As N cannot win, N would like to come 2nd. As O will move after N, N must come 2nd to O and thus allow O to win. For O to win, N must select a N option - there only needs to be one N option for this to happen. Note that N does NOT select an N option as the 'N' matches - N selects an N option for player O to win. □

Thus, putting all of this together, including games with winning strategies, N's option prorities in a game G go in the following order:

Theorem 7. In any game G, Player N chooses options in the following order:

$$P > Q > N > O$$

Proof. Each inequality has been proven in previous theorems and sections of this paper. □

In 3 player games, player N often has much control over the game's result, especially in Q games. Thus, by using these ideas with the podium rule, N can do his best to try and come 2nd, if not 1st place.