MAKER-BREAKER GAMES

ARCHI KANUNGO

Abstract. Maker-Breaker Games, is a kind of positional game usually played by two players Maker and Breaker. Positional games are finite games of complete information with no chance moves. In this paper, we will look at some important theorems related to winning strategies of players. Gradually, we will also develop some ideas related to the strategies of players in weak and strong games.

1. Introduction

Positional game is a kind of combinatorial game for two players. It is generally described by:

• X- It is a finite set of elements. Often X is called the board and its elements are called positions.

• $\mathcal{F}$- It is a family of subsets of X. A collection $\mathcal{F}$ of subsets of a given set X is called a family of subsets of X, or a family of sets over X. These subsets are the winning sets.

The Maker-Breaker Games is played as follows: Usually, Maker makes the first move where he wins, if he manages to occupy all the elements of the winning set; while Breaker wins if she manages to prevent this by occupying at least one element in winning set.

For example- If the winning sets are $\{\{1,2,3\},\{4,5,6\}\}$, then if Maker claims $\{\{1,4\},\{1,5\},\{1,6\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\}\}$ elements of winning set. Then Maker becomes the winner, if Breaker claims any one element from $\{\{1,4\},\{1,5\},\{1,6\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\}\}$, let's say $\{3,6\}$ then she is the winner.

To make communication easier, we typically refer to Maker as "he" who colors his elements "red" whereas Breaker is referred as "she" who colors her elements as "blue". Hypergraph is a generalization of a graph in which an edge can join any number of vertices. On a hypergraph, the elements of X are denoted by V (vertices) while elements of $\mathcal{F}$, for example: $\{\{1,2,3\},\{4,5,6\}\}$ are denoted by E (edges). Let $(V,\mathcal{F})$ be an finite hypergraph. So, H=$(V,\mathcal{F})$. Then, $V = V(H)$ is a finite set and $\mathcal{F} = \mathcal{F}(H)$ is a subsets of V. The two players alternatively occupy the elements, where each player claims one element per move. Maker wins if he takes all elements of some $A \in \mathcal{F}$, otherwise Breaker wins without any confusion. Normally Maker plays first but it's also possible for Breaker to play first.

Date: August,2021.

MAKER-BREAKER game is a weak positional game on hypergraph $(V, \mathcal{F})$. "Weak" has been coined to distinguish from "Strong" games. "Weak" states that the Breaker doesn't win if she claims an edge $A \in \mathcal{F}$ by herself. Here comes a question: Who can win? If Maker or Breaker has a strategy that always tends to win. A strategy includes mapping σ from finite sequences $(x_1, x_2, x_3, ...x_n)$ of distinct vertices of H to $V(H)/\ \{x_1, x_2, ..., x_n\}$, where $n < |V(H)|$. The x_i describes a course of play up to some point, then σ determines the next move. So, it is quite clear that Maker strategy σ is defined for sequences of even length whereas the sequences of odd length can be defined as Breaker's strategy.

A *winning strategy* is described as a strategy that makes a player win against all possible moves of its opponent. It's quite clear in combinatorial game theory where it tells that either player must have a *winning strategy*. So, draws are also impossible in MAKER-BREAKER games.

Definition 1.1. A hypergraph H is a winner if Maker, playing first, has a *winning strategy* on H, otherwise, when Breaker has a *winning strategy*, we call it a loser.

The computational complexity determines whether a given hypergraph is a winner.At each move, we could simply determine the value of the outcomes of all our options together with all possible opponent plays. From this we would then be able to tell which moves are the best. This simply sums up the definition stated above.

2. TIC-TAC-TOE ON HYPERGRAPH

TIC-TAC-TOE, is played by two players with symbols of X and O. The first player tries to place his symbol on squares of the 3x3 grid. This generally results in a line either in a row, column or a diagonal. If all the squares get occupied and no player has covered a line then the game results in a draw. Generally, the two players makes their best moves often termed as "perfect play" which definitely results in a draw.

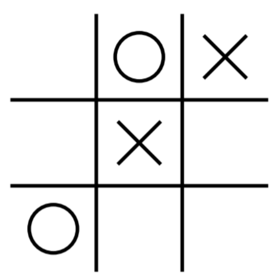

Figure 1. A classic game of Tic-Tac-Toe

When we consider TIC-TAC-TOE game on the same hypergraph $(V, \mathcal{F})$, then the two players alternatively occupy the elements of V, where each player claims one element per move. It has 9 vertices and 8 edges. The game is in favor of the player who occupy all the elements of the winning set $A \in \mathcal{F}$, otherwise the game ends in a draw. Just like in MAKER-BREAKER version, here also one player chooses one color and if he colours an edge monochromatically (have a single color), the player wins otherwise it results in a draw.

TIC-TAC-TOE can be played as a MAKER-BREAKER game.In this variant, Maker's goal is to pick 3 squares in a row, whereas goal of Breaker is to prevent Maker. Then, Maker's has a *winning strategy*. This is in contrast to the classical variant which is a strong positional game where second player has a *drawing strategy*. The difference between strong and weak games could be easily seen with the simple example of TIC-TAC-TOE. Everyone knows that it is a draw in strong version. Maker can win on the 3×3 board in the weak game because in certain situations Breaker lacks counter threats. This just results to a new strategy called *drawing strategy* which always leads to a draw at least. When we follow this strategy, we are sure that we will not loose and there is a chance that we may win the game.

REMARK- If the first player can win the strong game on a hypergraph H, Maker can win the weak game on H. If Breaker can win the weak game on a hypergraph H, the second player can force a draw in the strong game.

• We refer P1 as the first player and P2 as the second player. Perfect strategy refers to the strategy whose outcome (win, lose or draw) can be correctly predicted from any position, assuming that both players play perfectly. It leads to the best possible outcome for that player regardless of the response by the opponent.

Definition 2.1. A particular state of the game is called a position, it's a choice of two disjoint subsets (have no element in common) of V marked by P1 and P2 such that P1 has marked either one or zero more elements than P2. It happens when P1 and P2 both play with perfect strategy.

Definition 2.2. A *winning strategy* S for P1 is a function from the set of positions on V to the set of possible moves such that the game played in $(V, \mathcal{F})$ where P1 plays according to S, P1 will win. While the drawing strategy for P1 is the same. This results in a draw or P1 wins.

Proposition 2.3. *For any hypergraph on $(V, \mathcal{F})$, exactly one of the following holds:*

(1) *P1 has a winning strategy*

(2) *P2 has a winning strategy*

(3) *Both P1 and P2 have a drawing strategy*

Proof. We use induction to prove the proposition stated above. It begins with the proof of the base case, that is showing that any hypergraph on $(V, \mathcal{F})$ have a position that is P1 winning when P1 has a perfect strategy. Similarly, the same could be seen in P2 as well. Based upon that, it results in three outcomes- winning for P1, or P2, or a draw with m marked. Assuming the induction hypothesis, let $V = n$, then every position in which all n elements have three possibilities which include: winning for P1 or P2, or a draw. Let P be a position with $m - 1$ elements. Suppose P1 is next to move, then there are finitely many possible resulting positions after P1 moves. If any of these positions are P1 winning, then P is P1 winning. While if these positions are P2 winning, then P is P2 winning. In order to prove the inductive step, that is the initial position of V has no elements marked in either P1 winning, P2 winning, or drawing one assumes the inductive hypothesis that if neither position is P1 winning, nor P2 winning then it ends in a draw. Hence, by induction it's clear that for any hypergraph $(V, \mathcal{F})$, exactly one holds true that is either P1 has a winning strategy, or P2 has a winning strategy, or both have a drawing strategy. ■

Proposition 2.4. *No hypergraph game is a $P2$ win. In particular, if a hypergraph game can never end in a draw, then it is a $P1$ win.*

Proof. Here we use the technique of "Strategy Stealing". Let S be a *winning strategy* for $P2$. We will find a *wining strategy* for $P1$, yielding a contradiction. Note that an extra move can never hurt either player. $P1$ begins by playing anywhere, say $x \in V$. It's like $P1$ were the second player with an extra element marked. So, playing according to S is a win for $P1$. If at any point S says $P1$ should play at x, then $P1$ plays anywhere unmarked. Similarly, if it says $y \in V$ on its next turn resumes play according to S. Again if S wants $P1$ to play at y, then $P1$ plays anywhere unmarked and then resumes to S, and so forth. Hence, $P1$ has a winning strategy. But, then $P2$ has lost- contradicting the supposition that $P2$ had a guaranteed *winning strategy*.

■

• **Strategy Stealing Argument**- It's an argument proposed for many two-player games where the second player cannot have a guaranteed *winning strategy*. Whoever plays first in a strong hypergraph game can force atleast a draw.This applies to any symmetric game where the first player can "use" the second player's strategy in which an extra move can never a disadvantage.

Theorem 2.5. *In a strong game played on $(V, \mathcal{F})$. First player can at least guarantee a draw. [5]*

Proof. This proof applies the "Strategy Stealing Argument". Let us assume to the contrary that second player has a winning strategy S. Then we can convert S into a winning strategy for the first player. The first player makes its move at random, where it pretends to be the second player by "stealing" the second player's strategy. This results in a victory for the first player. If strategy S calls first player to move in the square that was chosen at random, then it moves there. However, this is contradiction as no winning strategy for the second player exists. ■

3. Strategies for Maker-Breaker Games

Maker-Breaker Games have some similarities to strong games. Maker can be compared to first player, while Breaker to the second player. However, these games are more different than similar. As, here draws are not possible. So, generally called as weak games. Let us closely take a look at strategies of each player. Let's start with the Breaker.Breaker's win is related to 2-coloribility in hypergraphs, by Erdős - Lovász theorem.

• Erdős-Lovász theorem: Let $\mathcal{F}= (A_1, A_2, A_3, ...)$ be an n-uniform hypergraph. Suppose that each A_i intersects at most 2^{n-3} other $A_j \in \mathcal{F}$ (local size). Then there is a 2-colouring of the 'board' $V{=}\bigcup_i A_i$ such that no $A_i \in \mathcal{F}$ is monochromatic. [3]

"Under the Erdős-Lovász theorem, Breaker has a blocking draw strategy, i.e., he can block every winning set in the weak game on $\mathcal{F}$". [2]

Proposition 3.1. *If the Maker-Breaker game played on a hypergraph $(V, \mathcal{F})$ is Breaker's win, then $\mathcal{F}$ is 2-colorable.*

Proof. Here, the two-player game on $\mathcal{F}$ where the first and second player think themselves as player and follows the Breaker's *winning strategy*. Due to this, each of them comes as a winner. Thus, it proves that $\mathcal{F}$ is 2-colorable. ■

Theorem 3.2. *Let $(V, \mathcal{F})$ be a hypergraph. If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{2}$$

, then $\mathcal{F}$ is Breaker's win. [4]

A potential based strategy guarantees Breaker a *winning strategy*. The potential of any winning-set A with $|A|$ unoccupied vertices as $2^{-|A|}$.So, the potential of a set occupied by Maker is indeed $2^-0 = 1$. Whenever Maker takes an element, the potential of every set containing it increases to $2^{-(|A|-1)}$,i.e., increases by $2^{-|A|}$. Whenever Breaker takes an element, the potential of every set containing it drops to 0 and decreases by $2^{-|A|}$. To every element, we assign a value equals to the total potential increase such that if Maker takes it, i.w., $w(v) := \sum_{v\in A} 2^{-|A|}$. The winning strategy of Breaker is to pick element with a highest value. If the potential at the first Breaker turn is less than 1, Breaker wins. In Maker's first turn, he can atmost double the potential by taking an element contained in all winning set. Therefore, it's clear that the potential is less than 1/2. Then, by Erdős-Selfridge theorem: If

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{2}$$

, then $\mathcal{F}$ is Breaker's win.

Definition 3.3. The chromatic number of hypergraph $(V, \mathcal{F})$ is the least integer $r \geq 2$ such that the points of V can be colored with r colors yielding no monochromatic $A \in \mathcal{F}$. If the chromatic number of $(V, \mathcal{F})$ is bigger than 2, then draw is impossible. In this case First-Player's drawing strategy is actually a *winning strategy*.

Theorem 3.4. *Suppose the board V is finite, and the family $\mathcal{F}$ of winning sets has chromatic number at least 3. Then first player has a winning strategy in the strong game on $(V, \mathcal{F})$.Then, the same strategy guarantees a win for Maker, as the first player, in the* Maker-Breaker *version on the same hypergraph. ("Ramsey Criterion") [1]*

The above theorem deals with a property through which we can easily diagnosis the winner without being able to say *how* one wins.

• **Copy Cat Pairing Strategy**- This strategy is just similar to "Strategy Stealing Argument". This works in the Maker-Breaker version which just shows an explicit version of "Strategy Stealing Argument" where Maker win even if he is the second player.

Theorem 3.5. *Let $(V, \mathcal{F})$ be a finite hypergraph of chromatic number at least 3, and let $(V', \mathcal{F}')$ be a point-disjoint copy of $(V, \mathcal{F})$. Let $W = V \cup V'$ and $G = \mathcal{F} \cup \mathcal{F}'$.*

(1) *Then Maker has an explicit winning strategy in the* Maker-Breaker *game on (W, G). [1]*

Proof. Let us assume Maker to be second player. Let $f : V \to V^{'}$ be the isomorphism between $(V, \mathcal{F})$ and $(V^{'}, \mathcal{F}^{'})$. Here Maker's winning strategy makes use of the copy cat stealing strategy. If Breaker's last move was $x \in V$ or $x^{'} \in V^{'}$ then Maker's next move will be $f(x) \in V^{'}$ or $f^{-1}(x^{'}) \in V$. As the chromatic number of $(V, \mathcal{F})$ is not less than 3, at the end one of the two players will completely occupy a winning set from $\mathcal{F}$. If the player is Maker then we are all done, but if Breaker occupies some $A \in \mathcal{F}$, then Maker occupies $f(A) \in \mathcal{F}^{'}$, then we are all set again. ■

References

[1] József Beck. Positional games and the second moment method. *Combinatorica*, 22(2):169–216, 2002.
[2] József Beck. Positional games. *Combinatorics, Probability and Computing*, 14(5-6):649–696, 2005.
[3] Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Colloquia Mathematica Societatis Janos Bolyai 10. Infinite and Finite Sets, Keszthely (Hungary)*. Citeseer, 1973.
[4] Paul Erdös and John L Selfridge. On a combinatorial game. *Journal of Combinatorial Theory, Series A*, 14(3):298–301, 1973.
[5] Michael Krivelevich. Positional games. *arXiv preprint arXiv:1404.2731*, 2014.