A Winning Strategy in the Angel and Devil Problem

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July 2021

1 Introduction

In this paper, I shall go over the proof of a wining strategy for the any Angel with power 2 or more in the Angel problem. The Angel problem is a question proposed by John Conway. The game is played by two players, usually referred to as the angel and the devil, on an infinite board. The angel has a power k specified at game start. The board starts empty with an angel on one of the squares. On each turn, the angel can move to an adjacent square(including diagonals) at most k times. The devil, on its turn, can destroy a square on the board so that the angel is no longer eligible to move there. The Devil wins if the Angel can no longer move, and the Angel wins by surviving indefinitely.

It was proven that, when k is 2, the Angel has a wining strategy. Just like when solving a maze puzzle, you can solve it by keeping your left hand on the wall the entire time, the Angel problem is somewhat similar. She divides the board into evaded squares, free squares, and blocked squares. At the beginning, she declares the Left half to be evaded, and right half to be free, and start moving up along the boundary of the free squares and the evaded squares. The boundary forms the path of the Angle, and is adjusted based on the Devil's move. The idea is that the Angel keeps trying to go up, but if the Devil puts an obstruction in the way, the Angel will (carefully) go around it. The tricky part of the proof is to show that the Devil can't force the Angle to go in a circle, because if so, she will be trapped.

2 Border Curves

A key concept in the strategy is *border curves*. We define a *segment* to be the border between two adjacent squares on the board. We consider continuous curve built from an infinite sequence of segments. The curves are directed from their *past* to their *future*. When considering a segment as part of a curve, it is also directed; otherwise, it is not.

A border curve partition the board into a *left* set and a *right* set. All the squares in the left set has to be connected, while some components in the right set can be enclosed in the left set.

Thus, we formally define a border curve below:

Definition 1: Let s be a segment in a border curve. The *left* square of s is the square on the left hand side of s and the *right* square of s is the square on the right hand side of s when looking into the future direction.

Definition 2: Let κ be a continuous and directed curve consists of infinite sequences. κ is said to be a *border curve* if there exist a set of squares V_κ such that all the following conditions are satisfied:

1) No segment exist more than twice in κ.

2) If a segment appears exactly once in κ, its left square is in V_κ and its right square is not.

3) If a segment appears exactly twice in κ, then they are in opposite directions, and both adjacent squares are in V_κ..

4) If a segment does not occur in κ, then either both of its adjacent squares are in V_κ or both are not in V_κ.

5) Both V_κ and its complement are infinite.

6) V_κ is connected. In particular, two squares are said to be neighbours if and only if they are adjacent to a segment that does not occur in κ.

We define V_κ to be the *left* set, ts complement to be the *right* set and κ traces the infinite border between them. Figure 1 shows an example of a border curve.

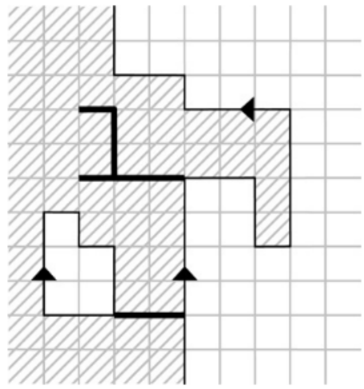

Figure 1: An example of a border curve. The arrows are directed towards the future, all the squares in V_κ are shaded. Segments that occur twice in κ are marked bold.

Lemma 1: *The left side and right side of a border curve κ are unique.*

We define two transformations on a border curve κ into another border curve:

- *Extension:* We can replace a segment s on κ with right square $q \notin V_\kappa$, replacing it with the other three segments that border q such that q is the left square of all of them.

- *Contraction:* If two consecutive segments in κ traverse the same segment on the board, we can delete both of them.

Lemma 2: *Let κ be a border curve, μ be an extension of κ involving square q, ν a contraction of κ. We have $V_\mu = V_\kappa \cup \{q\}$ and $V_\kappa = V_\nu$*

Definition 3: Let κ and ν be border curves. If ν can be obtained through a finite sequence of transformations from κ then ν is a *descendant* of κ.

Lemma 3: *If ν is a descendant of κ, $V_\kappa \subseteq V_\nu$*

Lemma 4: *Let κ be a border curve and s be a segment that occurs twice in κ. Let ν be the curve obtained by erasing all the segments between the two occurrences of s inclusively in κ. Then ν is a descendant of κ and therefore also a border curve.*

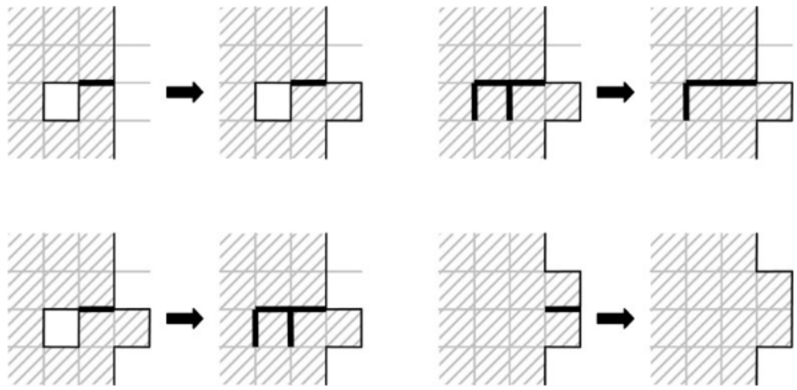

Figure 2: Left: examples of extension. Right: examples of contraction.

3 The Angel's Strategy

In the Angel's strategy, the angel maintains a *path* that represents his past movements and plans for the future. The path is always a border curve. A *perch* is a segment on the path and the Angel is in the right square of the perch. On his turn, the Angel would move the perch two segments along the path towards the future and would then move into its right square. It is obvious that such a move would never exceed the Angel's power.

At the start of the game the path is a infinite straight line that goes from South to North that contains the perch whose right square is the Angel's starting square. Each turn, the Angel will survey the board in the future part of the path to see if there is an sufficiently amount of blocked squares sufficiently close to the path. He will then update the path, which is a descendant from the current one, that evade those squares, therefore guaranteeing that the Angels can move along the path forever. This process will be formalized in the section.

Definition 4: At any time, the board is partitioned into *evaded, blocked* and *free* squares. The evaded set is the left set of the current path, the blocked set is all the squares on the right set of the path that is eaten by the Devil, and the free set is the set of the remaining squares.

Initially, the entire Western half of the board is evaded and the entire Eastern part is free. Every time the path is updated, some free and blocked squares may become evaded. By eating a free square, the Devil can convert it into a blocked square.

Definition 5: We define λ_i to be the path on the i'th turn after updating; λ_0 is the initial path.

Let κ be a descendant of λ_0. Since κ can be obtained from λ_0 through a finite number of transformations, they must be the same sufficiently far in the past and in the future. Thus, we can define L_κ, the *length* of κ, to be the number of segments in κ subtract the number of segments in λ_0 after removing infinite parts sufficiently far in the future and in the past from both curves.

Let j be a turn and κ be a border curve. We define $n_\kappa(j)$ to be the number of squares in V_κ that the Devil converted from free to blocked before turn j. It is important to understand that any such square is not free on turn j but was free at the time when the Devil converted it.

Lastly, we define p_i to be the perch after the Angel's move in turn i; p_0 is the initial perch.

As a shorthand, we will write $n_i(j)$ for $n_{\lambda_i}(j)$ and L_i for L_{λ_i}

Informally, the rule which the Angel updates his path can be stated as such: The future of the path is to be as short as possible, but the Angel can add segments to it if for every 2 segments added, an additional blocked square is evaded. Under these constraints, the Angel seek to evade as many blocked squares as possible.

As the Angel start turn i, λ_{i-1} is the current path and p_{i-1} is the current perch. We may now proceed to define the update rule formally. Let P_i^1 be the set of border curves μ that satisfy condition 1 and 2:

1. μ is a descendant of λ_{i-1}
2. μ is equal to λ_{i-1} in the past up until and including turn p_{i-1}

   Then, let P_i^2 be the set of $\mu \in P_i^1$ that satisfy condition 3:

3. For every $\kappa \in P_i^1$, we have $L_\mu - 2n_\mu(i) \leq L_\kappa - 2n_\kappa(i)$

   Finally, let P_i^3 be the set of $\mu \in P_i^2$ that satisfy condition 4:

4. For any $\kappa \in P_i^2$, $n_\mu(i) \geq n_\kappa(i)$

In other words, rule one and two states that the updated path is a descendant from the current path that only differ in the future, rule three states that for every two extra segment on the update path with respect to λ_0 one blocked square has to be evaded(since the minimum value of $L_\mu - 2n_\mu(i)$ is 0), and rule four states that the updated path should evade as much blocked squares as possible.

Lemma 5: *If i and j are turns and $j > i$, then $L_j - 2n_j(j) \leq L_i - 2n_i(i)$.*

Proof. Notice that $\lambda_{i-1} \in P_i^1$ and $\lambda_i \in P_i^2$. Through rule number three, we have

$$L_i - 2n_i(i) \leq L_{i-1} - 2n_{i-1}(i)$$

Since the Devil can not convert any more free squares into blocked squares in V_{i-1} after turn $i-1$, we have

$$n_{i-1}(i) = n_{i-1}(i-1)$$

Therefore,

$$L_i - 2n_i(i) \leq L_{i-1} - 2n_{i-1}(i-1)$$

and a little induction completes the proof. □

4 Proof that the Angel wins

Consider a future path segment s and its right square q. q can not be blocked since the update rule would have preferred to evade it. The other case is that it could be evaded, which means that there is a loop. The loop can be entirely in the future or it can include the perch, thus enclosing the angel. However the former case is not possible since the update rule will filter out all such curves and in the latter case the Angel would have detected the threat earlier and the path would have been updated to avoid the region entirely. Even if a loop is formed, it would be small enough that the Angel can jump out of it in the same turn. We now proceed to prove this formally:

Lemma 6: *Let s be a segment of λ_j in the future of p_j, and q be the right square of s. Then at the end of turn j, q is not blocked.*

Proof. We assume that q is blocked. Suppose μ is an extension of λ_j where s is replaced by the other three borders of q. Then $L_\mu = L_j + 2$ and $n_\mu(j) = n_j(j) + 1$, therefore $L_\mu - 2n_\mu(j) = L_j - 2n_j(j)$ and $n_j(j) < n_\mu(j)$. Since $\lambda_j \in P_j^2$, we have $\mu \in P_j^2$. But since $\lambda_j \in P_j^3$, rule 4 implies that $n_j(j) \geq n_\mu(j)$, thus we have a contradiction. $\square$

Lemma 7: *Let s be a segment of λ_j in the future of p_{j-1}, and let q be its right square. Assume that q is avoided at the end of turn j, then s is the very next segment of λ_j after p_{j-1}*

Proof. Definition 2 implies that s has to appear in λ_j twice. Let s_1 denote the first occurrence and s_2 denote the second occurrence. Define κ to be the curve obtained by deleting the part of λ_j from s_1 to s_2 inclusively. By Lemma 4, κ is a descendant of λ_j and its length is

$$L_\kappa = L_j - l \tag{1}$$

where l is the number of segments from s_1 to s_2, inclusively. By Lemma 3, $V_{\lambda_j} \subseteq V_\kappa$, therefore

$$n_j(j) \leq n_\kappa(j). \tag{2}$$

If s_1 is in the future of p_{j-1} then we have $\kappa \in P_j^1$. By rule number three, we need to have $L_j - 2n_j(j) \leq L_\kappa - 2n_\kappa(j)$ but that is impossible. So s_1 is in the past or coincide with p_{j-1}. Let i be the turn when the Angel moved the perch beyond or to s_1. p_i is at or after s_1 while p_{j-1} is before s_2. From p_i to p_{j-1} the Angel moves the perch at most $l-2$ segments, moving it 2 segments per turn. Therefore we have $l - 2 \geq 2(j - i - 1)$, or simply

$$2((j - i) \leq l \tag{3}$$

with equality only when p_i coincides with s_1 and s_2 is immediately after p_{j-1}. From Lemma 5 we have

$$L_j - 2n_j(j) \leq L_i - 2n_i(i) \tag{4}$$

and since the Devil can only eat $j - i$ squares between turn i and turn j, it follows that

$$n_\kappa(j) - n_\kappa(i) \leq j - i \tag{5}$$

Taking the sum of Eqs.(1) $+ 2 \cdot$ (2) $+$ (3) $+$ (4) $+ 2 \cdot$ (5) we get

$$L_\kappa - 2n_\kappa(i) \leq L_i - 2n_i(i) \tag{6}$$

In turn i, rule number three implies

$$L_\kappa - 2n_\kappa(i) \geq L_i - 2n_i(i)$$

The only way both can be true is when equality holds, that means that equality must hold in (3).

The proof is almost complete, we just need to show that s_1 is not too far in the past to the point where no such i exists. To show this, we just have to show that we can assume j to be arbitrarily late in the game. Suppose that for the first m turns we force the Devil to pass by only eating evaded squares. The Angel would trod along the path, until m turn passes. Due to the symmetry of the initial path, this game is equivalent to the original game regardless of the value of m. Thus, j can be any turn, no matter how late it is. □

Theorem 8: *The presented strategy permits the Angel to play indefinitely without ever landing on an eaten square.*

Proof. Let j be any turn and q be the right turn of p_j. At the end of turn j, q cannot be blocked by Lemma 6 and cannot be evaded by Lemma 7, thus it must be free. □