TWO KINDS OF NIM

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ABSTRACT. In this expository paper, we discuss two types of the combinatorial game **Nim**. Namely, we explore the original variant, and a variant known as **Fibonacci Nim**. We start the first section with a brief introduction on what a combinatorial game is, and soon thereafter move on to our main discussion in following three sections. For both Nim and Fibonacci Nim, we start our discussion by experimenting with the game, and we end by proving their respective winning strategies.

1. A Brief Introduction

What is a combinatorial game? Well, any game that satisfies the following requirements is a combinatorial game:

- There are two players, who we will denote as Left and Right
- There are several positions that can be reached throughout the game (note that having infinitely many positions is allowed in certain cases)
- There are well-defined rules that specify which moves Left and Right can make from each position
- Left and Right alternate taking turns.
- At all times during game play, both players know the current position and all the positions that can arise in the future; this is known as *complete information.*
- There are no chance elements involved.
- The player who is unable to make a move loses (this is known as the "Normal Play" convention. In games that follow the misère convention, the player who can make a move in the end loses instead).
- The game will always end after a finite number of moves. No draws exist.

Although we say that we can only have two players, what if we consider a game with three? The problem with three player is that one player cannot usually guarantee a win against the collusion of the other two players.

We do not want games to last forever, and a game in which a position can be repeated is called a *loopy* game. To avoid being *loopy* some games impose extra rules, like forbidding previous positions to be repeated. These rules are called *ko* rules. Note that there do exist games with infinitely many positions, but which does not last for infinitely many moves. It might happen that there is some integer N so that any game sequence lasts at most N moves, or it might happen that games can last for arbitrarily many moves, but still necessarily end after a finite number of moves.

We now know what a combinatorial game is, but is there anyway we can find the ending outcome of a game? The answer to this question is yes – the outcome of a combinatorial game must fall in 4 categories, namely:

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- L, which denotes that Left wins regardless of who starts
- R, which denotes that Right wins regardless of who starts
- N, which denotes that Left wins if Left starts and Right wins if Right starts
- P, which denotes that Right wins if Left starts, and Left wins if Right starts

2. What is an impartial game?

An **impartial game** is a combinatorial game in which all of the players have the same moves available to them. For example, the combinatorial game of **Red-Blue Hackenbush** would not be an impartial game, since one player removes blue edges, and another player removes red edges (we do not explore Red-Blue Hackenbush in the paper, but encourage the reader to explore it by themselves). On the other hand, the game of **Green Hackenbush** would be considered an impartial game, since both players can only remove Green edges (again, we do not explore this game in this paper). Also, there are only two outcome classes for impartial games. Namely, N and P. This is because winning is solely based on who moves first or second, since all moves are available to both players.

3. The game of Nim

Nim is an impartial combinatorial game played by two players. There are initially several (or one) piles of stones, and each player moves by taking at least one stone from a single pile. The game follows the Normal Play convention, so the player with no moves left loses. There is no luck in this game – there is a winning strategy that will work every time.

It is easy to analyze basic positions in Nim. For example, say we have two piles with one stone in each. Clearly, the second player will win, because player one will be forced to take on coin, and player two can just the the remaining one. If, instead, we have one pile with two stones and another pile with one stone, the first player will always be able to win – Player one should take one stone from the pile with two, which will reduce the game to the one we analyzed before this. After this, player one is guaranteed to win for the same reasons as in the other argument.

We can extend this pattern or "recursion" even further. [2] If there are two piles with two stones each, the second player is guaranteed to win, since if the first player takes one stone, we reduce this problem to the second scenario we analyzed, and if the first player takes two stones, the second player can take the other pile and force a win.

These analyses hint to us that there must be a (relatively simple) winning strategy, and indeed there is. The strategy goes as follows:

- Write the size of the heap(s) in binary.
- Sum these number using **Nim addition**. (Nim addition the same as regular binary addition, but without carrying. We denote the Nim sum between two piles of size x_1 and x_2 as $x_1 \bigoplus x_2$). Your result is known as the **Nim sum** of your game.
- If the Nim sum at the beginning of the game is 0, then the first player has a winning strategy. Player one must simply make a move, such that the Nim sum after their move is 0. Any move player two makes will result in the Nim sum shifting away from 0, in which case player one must make the Nim sum 0 again. The guarantees player one a win, since if player two could win, they would have to make a move that results in a Nim sum of 0 (which is the same as a move that leaves no coins left over), which is impossible.

- If the Nim sum at the start of the game is not equal to 0, then player two will have a winning strategy. When player one makes the first move, the Nim sum will no longer be 0. After this, player two can make the Nim sum 0 again, and ensure a win much in the same way as the last argument.

The crux of this argument lies in two key claims. Which are

- If the Nim sum is a certain position of the game is 0, making any move will alter the Nim sum.
- If the Nim sum in a certain position of the game is not 0, there is always a move that can make the Nim sum 0.

Here is a proof of the first claim:

Proof. Let us have n piles of stones with $x_1, x_2, x_3, ..., x_n$ stones in each. and let the Nim sum of these piles be denoted s. Let t be be the sum of the heaps y_i after the move t=$y_1 \bigoplus y_2 \bigoplus y_3 \bigoplus ... \bigoplus y_n$. Then if $s = 0$, then the next move causes some $x_k \neq y_k$ and the rest of the $x_i = y_i$ for $i \neq k$, since only one pile is changed. Thus:

$t = 0 \bigoplus t$
$= s \bigoplus s \bigoplus t$
$= s \bigoplus (x_1 \bigoplus x_2 \bigoplus ... \bigoplus x_n) \bigoplus (y_1 \bigoplus y_2 \bigoplus ... \bigoplus y_n)$
$= s \bigoplus (x_1 \bigoplus y_1) \bigoplus (x_2 \bigoplus y_2) \bigoplus ... \bigoplus (x_k \bigoplus y_k)$
$s \bigoplus x_k \bigoplus y_k$.

If $s = 0$ then t must be nonzero, since $x_k \bigoplus y_k$ will never be 0. Thus, if one player makes the Nim sum 0 on their turn, the other player must make it nonzero on theirs.■

Here is a proof of the second claim:

Proof. Let d be the position of the most significant bit in s (remember that s is written in binary). Now choose a heap x_k such that it most significant bit is also in position d (one must always exist, the most significant bit of s must come from the most significant bit of any of the Nim heaps). Now choose to make the new value of the heap $y_k = s \bigoplus x_k$ by removing $x_k - y_k$ stones from the heap. The new Nim sum is

$t = s \bigoplus x_k \bigoplus y_k$ (from above)
$= s \bigoplus x_k \bigoplus x_k \bigoplus s$
$= s \bigoplus s \bigoplus x_k \bigoplus x_k$
$= 0$.

Hence we are done. [1] ■

4. Fibonacci Nim: An Introduction

Fibonacci Nim is a variant of Nim which is played as follows: We have two players removing stones from certain pile(s). On the first move, a player cannot take all of the coins, a player must take at least one stone each turn, and on each subsequent move, the number of coins a player can remove is at most twice the number removed in the previous move. Using the normal play convention, the player without a move loses.

For our purposes, we will only be analyzing Fibonacci Nim with one pile, since there is no known complete winning strategy for the game with multiple piles.

First of all, let us just explore and experiment with the game. Say we have one pile with 10 stones. Here is one way how the game may play out:

- Player one takes 2 stones.
- Player two takes 3 stones (which is allowed since $3 \leq 2 \times 2 = 4$).

- Player one takes 1 stone (which is allowed since $1 \leq 3 \times 2 = 6$)
- Player two takes 2 stones (which is allowed since $2 \leq 2 \times 2 = 4$).
- Player one takes 2 stones and wins the game (which is allowed since $2 \leq 2 \times 2 = 4$).

Playing through several more games like this, we realize that if on one player's turn there are 3, 5, or 8 stones left, that they will lose (as long as they can't take all 3, 5, or 8 left). But what do these number have in common, and what is the next number for which one player is guaranteed to lose?

The answer to both of these questions lie in the Fibonacci numbers. That is, 3, 5, and 8 are all Fibonacci numbers, and the next number for which one player is guaranteed to lose is 13, the Fibonacci number after 8.

In fact, if a player removes enough counters such that a Fibonacci number of them are left over, then that player must be able to win. Getting to a Fibonacci number of stones requires skill though, since the opponent must not be able to take all the coins and win the game. This fact is virtually the wining strategy for this game. We will now prove this winning strategy (the proof is entirely due to Misha Lavrov):

Proof. We need to use Zeckendorf's theorem, which states that any number can be written uniquely as the descending sum of non-adjancent Fibonacci numbers, to prove this winning strategy. We will not cover the proof of Zeckendorf's theorem here, but do encourage exploration of this fascinating theorem. Also, we take for granted the property that, for any $n \geq 2$, that $F_{n+2} > 2F_n$, which can be proven by induction. We represent a position in Fibonacci nim as an ordered pair (p,q), where p is the largest number of stones that can be removed, and q is the number of stones that are left. A starting positions with k coins can be represented as $(n-1, n)$.

Claim. A position (p, q) is a win for the first player if, in the Zeckendorf representation of q, the smallest Fibonacci number is at most p. (Call such positions *comfortable* to make them easier to refer to.)

Proof. All positions with $p \geq q$, in which the first player can take all the remaining coins, are obviously comfortable, so the claim is true in that case. It's enough to prove that:

From every comfortable position, some move can produce an uncomfortable position. From every uncomfortable position, all moves lead to comfortable positions. (In other words, to produce an uncomfortable position, we must be in a comfortable position.)

Statement 1 holds because if q has Zeckendorf representation $F_{a_k} + ... + F_{a_1}$, with $a_{i+1} \geq a_i + 2$ and $F_{a_1} \leq p$, we can take away F_{a_1} coins. Then the smallest Fibonacci number in what's left is F_{a_2}, the limit is $2F_{a_1}$, and $F_{a_2} > 2F_{a_1}$.

Statement 2 holds because, if we move from (p, q) to $(2x, q-x)$ where the smallest Fibonacci number in the Zeckendorf reprentation of $q - x$ is bigger than $2x$, then we can concatenate the Zeckendorf representations of $q - x$ and x to get a Zeckendorf representation of q. So if we move to an uncomfortable position, we must have done so by taking away the sum of the last few Fibonacci numbers in the Zeckendorf representation of q. In particular, we took away a number at least as large as the smallest Fibonacci number in the Zeckendorf representation of q, so we must have been in a comfortable position.

So the optimal strategy, starting from a comfortable position, is to always put one's opponent into an uncomfortable position. Whatever move one makes, their opponent will be forced to give them an uncomfortable position back, and this repeats until we are in a position where we take all the coins. ■

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References

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