Sylver Coinage

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1 Introduction

Sylver Coinage is a combinatorial game played by two players where the players take turns naming numbers that cannot be expressed as sums with non-negative multiples of numbers previously named.

For example, if 3 and 5 have already been named,

$$3, 5, 3 + 3 = 6, 3 + 5 = 8, 3 + 3 + 3 + 5 + 5 = 19$$

are some of the numbers that cannot be named. The player who names 1 loses.

2 Opening Positions

We can immediately see that some opening positions are bad. By the definition of the game, if you name 1 at any point in the game, you lose. If you name 2, then your opponent can reply with 3 if it is still available, and then all larger numbers will be unavailable. Similarly, if you name 3, then your opponent can reply with 2. Then by the same reasoning, all larger numbers will be unavailable. Hence, 1, 2, and 3 are bad starting moves, and bad moves in general.

Now suppose you start with 4 and your opponent replies with 5. Then it is easy to check that the only remaining numbers will be $\{1, 2, 3, 6, 7, 11\}$. Suppose your next move is a 6. Then your opponent can respond with 7 to leave you with a choice of 1, 2, or 3. Similarly, if your next move is a 7, your opponent can respond with 6 and leave you with 1, 2, and 3. If your response is 11, however, you will force your opponent to respond with 1, 2, 3, 6, or 7, at which point you can respond with a move that will guarantee your win. Hence, $\{4, 5, 11\}$ is a P-Position. From similar reasoning, we can determine that $\{2, 3\}, \{4, 6\}, \{6, 9\}, \{8, 12\}$ are P-positions, and that $\{1\}, \{2\}, \{3\}, \{4\}, \{6\}, \{8\}, \{9\}, \{12\}$ are N-positions.

R.L. Hutchings proved that if a and b are relatively prime, and $\{a, b\} \neq \{2, 3\}$, then $\{a, b\}$ is an N-position. From this, he was able to deduce his p-theorem, which states that if $p \geq 5$ is a prime number, then $\{p\}$ is a P-position. This led to his n-theorem, which states that if n is a composite number not of the form $2^a 3^b$, then $\{n\}$ is an N-position.

3 Clique Technique

Cliques have the property that any reply to a clique member must also be a clique member, and these two numbers must together exclude all numbers outside the clique. For example, in the discussion of the starting pair $\{4, 5\}$, the cliques are $\{1\}, \{1, (2, 3)\}, \{1, (2, 3), (6, 7)\}, \{1, (2, 3), (6, 7), 11\}$. From here, we can easily see that 11 is a good response to $\{4, 5\}$, causing $\{4, 5, 11\}$ to be a P position.

4 Strategy Stealing

Hutchings proved his main theorem with strategy stealing - given that t is the largest number not excluded by $\{a, b\}$, then if t is not a good response, then some other number is. We will call $\{a, b\}$ an end position. Note that any available number less than t will exclude t with a and b.

If t is a good response, then $\{a, b\}$ is an N-position. If t is not a good reply, then there is some number s that is a good response to $\{a, b, t\}$. Then s is also a good response to $\{a, b\}$, since s will exclude t. Hence, in general, an end position is an N-position if its t value is greater than 1.

5 Quiet Ends

Given a position $\{a, b, c, \dots\}$ with t being the largest value that is not excluded and s any non-excluded value less than t, we say that s quietly excludes t if t can be written as the sum of any number of $a, b, c, \dots$ and one s. A *quiet end position* is an end position in which the largest legal move is quietly excluded by every number that is legal.

The quiet end theorem states that if a is relatively prime to b and b_1, then $S = \{a, bc, bd, be, \dots\}$ is a quiet end position iff $S_1 = \{a, b_1c, b_1d, b_1e, \dots\}$ is. From this, we can deduce that $\{7, 9, 12\}$ and $\{7, 15, 20\}$ are quiet end positions because $\{7, 3, 4\}$ is. And in particular, these are end positions, so they are N-positions.