COPS AND ROBBERS

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1. Introduction

Cops and Robbers is a conceptually simple game, that however is not yet fully mathematically understood. In the game of Cops and Robbers, there are some number of cops who are chasing a robber on a connected, finite graph, G. The cop(s) win if a cop can position itself on top of a robber, and the robber wins if he can avoid the cops forever. The cop(s) will be placed first on vertices of G, and then the robber will be placed on a vertex of G. The cop-side and the robber-side will alternate turns. Each piece may be moved once per turn, and only to an adjacent vertex. This paper explores some general theory behind Cops and Robbers, and then focuses more in depth on games that can be won with one Cop only. This paper focuses on the passive version of the game, where not only the cop(s) but also the robber can choose not to move during its turn.

Remark 1.1. The Cops will be abbreviated as C and the Robbers will be abbreviated by R.

Definition 1.2. The Cop number of a graph is the minimum number of cops required to catch a robber on G. It is written as C(G).

Definition 1.3. The closed neighbor set of a point A on a graph, denoted $N[A]$, is the set of vertices connected to A by an edge, and A.

Definition 1.4. A cop win graph is a graph with cop number 1. A n-cop win graph is a graph with cop number n.

Definition 1.5. A pitfall is a point A such that there is a point B where $B \in N[A]$, and $N[A] \subseteq N[B]$. Point A is the pitfall, and point B is a dominating vertex. $B \neq A$

2. Cop win Games

Theorem 2.1. *Complete graphs are cop win (where complete graphs are graphs that every vertex is connected to every other vertex).*

Proof. As wherever you put the cop, he can immediately move to the robber, he always wins. ■

Theorem 2.2. *All trees are cop win graphs.*

Proof. The cop can always move towards the robber and trap the robber in a decreasing amount of space. Within a finite amount of moves, The robber will be caught. ■

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3. Analysis of cop win graphs

Proposition 3.1. *If a graph is a cop win graph, then there must be a pitfall, or the graph is a point.*

Proof. Since the graph is a cop win graph, there must have been some moment in time such that there was a cop adjacent to the robber, and the robber had nowhere to go where the cop couldn't. So, $N(R) \subseteq N(C)$, and by definition R is at a pitfall. ■

Theorem 3.2. *If A is a pitfall, and B is A's dominating vertex, the game G has a cop number n if and only if after removing A and all edges with A as a vertex, the remaining graph has a cop number n.*

Proof. If the graph without A has a cop number of n, then in the original graph when the robber is not in A, proceed with the winning strategy with n cops. If the robber is in A, then play the winning response in G-A as if the robber had moved to B. Since then the robber has fewer moves than if he was actually in B, you will still win. That says that if you have a cop number of n after removing A, you win with n cops with A. So, the cop number of G≤G-A. If the graph with A has a cop number of n, then in the game without A you play as normal. If in the original graph the strategy was to move to A, you might as well move to B since then you have the same amount or more moves. ■

Theorem 3.3. *Any graph G is winning for one cop if and only if by removing pitfalls it can be reduced down to a point.*

Proof. By the previous theorem, we see that removing a pitfall does not affect the game in the slightest. After removing all pitfalls, and then removing more pitfalls, and so on you reach a game with no pitfalls, and the same cop number. A game with no pitfalls, and a cop number of 1, must be a single point because otherwise there would be no way for the cop to win. ■

Theorem 3.4. *Each cop win graph can number their vertices* $[n] = \{1, 2, ..., n\}$ *such that the vertex* 1 *is a pitfall, and when* 1 *is removed,* 2 *is a pitfall, and so on.*

Proof. By **Theorem 3.1**, you must have a pitfall in a cop win graph. You name a random pitfall 1, and remove it. Then, you can name another random pitfall 2, and remove it as well. By **Theorem 3.3**, we will eventually reach a single point, and we name it n. ■

4. Non-cop win Games

Theorem 4.1. *Circles of size more than 3 have cop number of 2.*

Proof. A circle of more than 3 can not be a cop win game because the robber can continue to run away from the single cop. Wherever the cop is, the robber will start two vertices away, and keeps moving in the same direction as the cop. However, adding a second cop basically changes the graph to a tree. Put one cop down the other can work the rest of the graph as if it is a tree anc catch the robber. If there are three points in the circle, just move directly to the robber. Two points, and again move directly to the robber. One point, and automatically captured. ■

Theorem 4.2. *Regular non-complete graphs have a cop number greater than 1 (where regular graphs are graphs where each vertex is connected to the same amount of vertices.*

Proof. For there to be a pitfall then two vertices have to connnect to the exact same vertices. If they do, then remove it. ■

5. Outer-planar Graphs

Definition 5.1. A graph is outer-planar if and only if it satisfies three requirements.

(1) All of the points fall on a circle.
(2) All edges either connect two adjacent points on the circle or a chord across the circle.
(3) If any two of the chords intersect, they do so at a vertex.

Theorem 5.2. *If there exists two vertices connected, then to get to a point in between them from the other side, you need to go onto one of them.*

Proof. Since no line can intersect, then to pass they must access a point on the other side, and that is only possible if that point is connected to one of the two vertices that were connected. ■

Theorem 5.3. *All outer-planar graphs have $C(G) \leq 2$.*

Proof. Strategy: The two cops are placed such that one of them splits the graph in half as perfectly as possible, and the other one is placed randomly. When the gets placed down, then the random cop starts moving towards it. Once the cop gets in the right half, then we re-arange the graph such that the restricted area becomes it's own circle, and the rest gets ignored. ■

6. General Graphs

Definition 6.1. A more expanded version of pitfalls is that you have a pitfall, A, and a set of dominating vertices $B, C, D...$ such that $N(A) \subseteq N(B) \cup N(C) \cup N(D)...$.

Theorem 6.2. *In a n-cop win graph, there must be a pitfall with n dominating vertices.*

Proof. Since you can win with n cops, there must be some point where a robber is trapped using at most n cops. If there is less than n cops, then you can just add irrelevant cops. ■

Theorem 6.3. *A graph has a cop number of the minimum amount of cops necessary to remove pitfalls, and end with a point.*

Proof. ■

7. Problems

(1) Find a graph such that C(G)=5.
(2) Prove that there are n-vertex graphs which require at least Ω ($\sqrt{}$n) cops.
(3) Ω (n) means the total number of prime factors of n counting prime factors with multiplicity.
(4) Prove that C(G)$\leq$O($\frac{n}{logn}$). The log is binary.
(5) Prove that computing the cop number of a graph is NP-hard.