On Riemann's 1859 Paper

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1 Summary

Let s be a complex variable.

Both sides of the equation

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

converge absolutely only in the half-plane $\Re s > 1$.

By analytic continuation, the function $\sum_{n=1}^{\infty} \frac{1}{n^s}$ can be extended to the whole complex plane $\mathbb{C}$ as a meromorphic function, $\zeta(s)$, with only a simple pole at $s = 1$, with residue 1, by the following equation

$$\zeta(s) = \frac{i}{2\sin(\pi s)\Gamma(s)} \int_{\mathcal{H}} \frac{(-x)^{s-1}dx}{e^x - 1}$$

where $\mathcal{H}$ is the Hankel contour extending from $+\infty$, around the origin once (but not around any point $\pm 2n\pi i, n \in \mathbb{Z}\backslash\{0\}$) and where we have chosen the principal branch of the logarithm function by making a branch cut along the positive real axis and such that $\log(-z)$ is real when z is real and negative(or equivalently, by making a branch cut along the negative real axis and such that $\log z$ is real when z is real and positive).

$\zeta(s)$ is then called the Riemann zeta function.

By the calculus of residues, the functional equation of $\zeta(s)$ is revealed to be

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s)$$

Then if

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-\frac{s}{2}}$$

it follows that $\xi(s)$ is an entire function and that it satisfies the functional equation

$$\xi(s) = \xi(1-s)$$

We have also

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), T \to \infty$$

where $N(T)$ denotes the number of roots of ξ with imaginary part between 0 and T, which is useful in the proof that

$$\xi(s) = \xi(0)\prod_{\rho}\left(1 - \frac{s}{\rho}\right)$$

where ρ runs over all the non-trivial zeros, (i.e. those such that $0 < \Re(\rho) < 1$) of $\zeta(s)$, counted with multiplicity and arranged in increasing order of moduli.

Let

$$F(x) = \frac{\pi(x+0) + \pi(x-0)}{2}$$

where

$$\pi(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} 1$$

Then if

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} F(x^{\frac{1}{n}})$$

we have

$$\frac{\log \zeta(s)}{s} = \int_0^{\infty} f(x) x^{-s-1} dx, \Re(s) > 1$$

By a form of Fourier's inversion theorem, we get when $a > 1$ if $s = a + bi$,

$$f(y) = \frac{1}{2\pi i} \int_{a-\infty i}^{a+\infty i} \frac{\log \zeta(s)}{s} y^s ds$$

or by integrating by parts to ensure convergence when we will replace the product representation of the Riemann Zeta Function

$$f(x) = \frac{1}{2\pi i \log x} \int_{a-\infty i}^{a+\infty i} \frac{d}{ds}\left(\frac{\log \zeta(s)}{s}\right) x^s ds$$

after which we recover $F(s)$ from $f(x)$ by a form of Möbius inversion

$$F(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{\frac{1}{n}})$$

1.1 Analytic Continuation and the Riemann Xi function

Proposition 1. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely for $\Re(s) = \sigma > 1$. It also converges uniformly in every closed half-plane $\Re(s) \geq 1 + \epsilon, \epsilon > 0$.

Hence, it is holomorphic in the open half-plane $\Re(s) > 1$.

Proof.

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty}\frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}}$$

so that, by the Weierstrass M test, $\sum_{n=1}^{\infty}\frac{1}{n^s}$ is uniformly convergent for $\Re(s) \geq 1+\epsilon, \epsilon > 0$ and absolutely convergent for $\Re(s) > 1$.

Now let C be any simple closed path in the half-plane $\Re(s) \geq 1+\epsilon, \epsilon > 0$. Then, for any positive integer K,

$$\sum_{n=1}^{k}\frac{1}{n^s}$$

is holomorphic for $\Re(s) \geq 1+\epsilon$, so that, by the Cauchy-Goursat theorem, we have

$$\oint_C \sum_{n=1}^{k}\frac{1}{n^s}ds = 0$$

Then, by the uniform convergence,

$$\lim_{k\to\infty}\oint_C \sum_{n=1}^{k}\frac{1}{n^s}ds = \oint_C \lim_{k\to\infty}\sum_{n=1}^{k}\frac{1}{n^s}ds = \oint_C \sum_{n=1}^{\infty}\frac{1}{n^s}ds$$

The uniform convergence of $\sum_{n=1}^{\infty}\frac{1}{n^s}$ also implies its continuity in the simple connected domain $\Re(s) \geq 1+\epsilon, \epsilon > 1$, so that, by Morera's theorem, it follows that $\sum_{n=1}^{\infty}\frac{1}{n^s}$ is holomorphic for $\Re(s) > 1$. □

Proposition 2. *For $s \in \mathbb{R}, s > 1$, we have*

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}dx$$

where we define the Gamma function, for the time being, to be

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}, s \in \mathbb{R}, s > 0$$

and by the recurrence

$$\Gamma(s) = \frac{1}{s}\Gamma(1+s)$$

Proof.

$$\frac{1}{n^s}\Gamma(s) = \int_0^\infty e^{-nx}x^{s-1}dx$$

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty}\int_0^\infty e^{-nx}x^{s-1}dx$$

For some $\epsilon > 0$, let $x \geq \epsilon$. Then for $n \in \mathbb{Z}$ we have

$$|e^{-nx}| = e^{-nx} \leq e^{-\epsilon x}$$

Moreover, $\sum_{n=1}^{\infty} e^{-\epsilon n}$ converges so that, by the Weierstrass M test, it follows that $\sum_{n=1}^{\infty} e^{-nx}$ is uniformly and absolutely convergent for $x \geq \epsilon$. In addition, e^{-nx} is continuous for $x \geq \epsilon$. Hence,

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-nx} x^{s-1} dx = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nx} x^{s-1} dx$$

We complete the proof by summing over the geometric series. □

Proposition 3.

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

is holomorphic for $\Re(s) = \sigma > 1$.

Proof. Let $\epsilon > 0$ and $\gamma > 1$ and suppose $1 + \epsilon \leq \sigma \leq \gamma$.

$$\left| \frac{x^{s-1}}{e^x - 1} \right| \leq \frac{x^{\sigma-1}}{e^x - 1}$$

$$\int_0^{\infty} \frac{x^{\sigma-1}}{e^x - 1} dx = \Gamma(\sigma)\zeta(\sigma)$$

so that, by the Weierstrass M test for integrals, we have that $\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$ is uniformly and absolutely convergent in every region $1 + \epsilon \leq \sigma \leq \gamma$.

Now

$$\frac{x^{s-1}}{e^x - 1} = \frac{e^{(s-1)\log x}}{e^x - 1}$$

is holomorphic and hence both continuously differentiable (by Cauchy's Theorem) and continuous with respect to s for $x > 0, 1 + \epsilon \leq \sigma \leq \gamma$. Moreover

$$\begin{aligned} \int_0^{\infty} \left| \frac{x^{s-1} \log x}{e^x - 1} \right| dx &\leq \int_0^{1} \left| \frac{x^{s-1} \log x}{e^x - 1} \right| dx + \int_1^{\infty} \left| \frac{x^{s-1} \log x}{e^x - 1} \right| dx \\ &= \int_0^{1} \frac{x^{\sigma-1} \log x}{e^x - 1} dx + \int_1^{\infty} \frac{x^{\sigma-1} \log x}{e^x - 1} dx \end{aligned}$$

The first integral is easily seen to converge as $\lim_{x \to 0} \dfrac{x^{s-1} \log x}{e^x - 1} = 0$ by l'Hôpital's rule. The convergence of the second integral is seen easily by replacing $\log x$ by x. Thus, by the Weierstrass M test, we have that

$$\int_0^{\infty} \frac{x^{s-1} \log x}{e^x - 1} dx$$

is uniformly convergent in the region $1 + \epsilon \le \sigma \le \gamma$. Since in addition it has been shown that $\frac{x^{s-1}}{e^x-1}$ is continuous and continuously differentiable for $x > 0$ in that same region, it follows that

$$\frac{d}{ds}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx = \int_0^\infty \frac{x^{s-1}\log x}{e^x - 1}dx$$

and so

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx$$

is holomorphic for $\Re(s) > 1$. □

Proposition 4. *The gamma function defined by*

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx, \Re(s) > 0$$

can be extended to a meromorphic function on the whole complex plane with only simple poles, and these at the non-positive integers only.

Proof. Firsly, we show that $\Gamma(s)$ is holomorphic in the open half-plane $\Re(s) > 0$.

Let

$$\Re(s) = \sigma > 0$$

Then

$$|x^{s-1}e^{-x}| \le x^{\sigma-1}e^{-x}, x > 0$$

But

$$\int_0^\infty x^{\sigma-1}e^{-x}dx$$

is convergent for $\sigma > 0$. Hence,

$$\int_0^\infty x^{s-1}e^{-x}dx$$

converges absolutely and uniformly in every closed half-plane $\Re(s) \ge \epsilon, \epsilon > 0$ by the Weierstrass M test.

Since moreover $x^{s-1}e^{-x}$ is continuous for $x > 0, \sigma \ge \epsilon$, then if C is any closed curve on the simply connected domain $\Re(s) \ge \epsilon$,

$$\oint_C \int_0^\infty x^{s-1}e^{-x}dxds = \int_0^\infty \oint_C x^{s-1}e^{-x}dsdx$$

$$\therefore \oint_C \Gamma(s)ds = \int_0^\infty e^{-x}\oint_C x^{s-1}dsdx$$

Since x^{s-1} is holomorphic on $\Re(s) \ge \epsilon, x > 0$, we have by the Cauchy-Goursat theorem

$$\oint_C x^{s-1}ds = 0$$

$$\therefore \oint_C \Gamma(s)ds = 0$$

Finally, since $\Gamma(s)$ is continuous on the simply connected domain $\Re \geq \epsilon$ (the integral being uniformly convergent and its integrand continuous), it follows by Morera's theorem that $\Gamma(s)$ is homolorphic for $\Re(s) > 0$.

Now, on making use of the functional equation

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

and

$$\Gamma(s) = \frac{\Gamma(s+m)}{s(s+1)\dots(s+m-1)}$$

successively for each positive integer m, we are able to extend the definition of $\Gamma(s)$ by analytic continuation to the whole complex plane as a meromorphic function with a simple pole at each non-positive integer. □

Proposition 5. *We have, for* $\Re(s) > 1$,

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}$$

Proof. Both sides agree of the domain $\Re(s) > 1$. Furthermore both sides are holomorphic for $\Re(s) > 1$. By analytic continuation, the proposition follows. □

Proposition 6. *The gamma function has no zeros.*

Proof. By considering

$$\oint_C \frac{z^{p-1}}{1+z} dz, p \in \mathbb{R}, 0 < p < 1$$

where C is a keyhole contour about the branch point $z = 0$, we easily deduce that

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin(p\pi)}, 0 < p < 1$$

Now when we perform the substitution $x \to \frac{y}{1-y}$ and make use of the known properties of the beta function, we deduce that

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}, 0 < p < 1$$

But now $\sin z\pi$ is an entire function with simple zeros at the integers so that $\frac{\pi}{\sin p\pi}$ is meromorphic with simple poles at the integers, and with no zeros; since both sides of our equation are meromorphic with the same poles and agree on a domain with limit points, it follows that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, s \in \mathbb{C}\backslash\{\mathbb{Z}\}$$

and so the left side has no zeros. □

Proposition 7. *Let $\mathcal{H}$ be the Hankel contour which consists of three parts: a ray above the positive real axis extending from $\Re x = \rho = +\infty$ to $\rho = \delta, \delta > 0$; a circular arc C of radius ϵ; a ray below the positive real axis extending from $\rho = \delta$ to $x = +\infty$. Then*

$$\int_{\mathcal{H}} \frac{(-x)^{s-1}}{e^x - 1} dx$$

is an entire function of s.

Proof. Recall that we have made a branch cut along the positive real axis and defined $\log(-z)$ to be real when z is real and negative.

$$\int_{\mathcal{H}} \frac{(-x)^{s-1}}{e^x - 1} dx = \int_{\infty}^{\delta} \frac{e^{(\log(x) - \pi i)(s-1)}}{e^x - 1} d\rho + \int_C \frac{(-x)^{s-1}}{e^x - 1} dx + \int_{\delta}^{\infty} \frac{e^{(\log(x) + \pi i)(s-1)}}{e^x - 1} d\rho$$

Considering firstly $\int_C \frac{(-x)^{s-1}}{e^x-1} dx$, let $\Re(s) = \sigma$; then since

$$\left| \frac{(-x)^{s-1}}{e^x - 1} \right| \leq \frac{(-x)^{\sigma-1}}{e^x - 1}$$

and since $\int_C \frac{(-x)^{\sigma-1}}{e^x-1} dx$ is clearly convergent, therefore

$$\int_C \frac{(-x)^{s-1}}{e^x - 1} dx$$

is absolutely and uniformly convergent in a closed disk $|s| \leq M$ for any $M > 0$. Then, since $\frac{(-x)^{s-1}}{e^x-1}$ is an entire function of s on C, therefore if γ is any simple closed curve in the complex s-plane, we have, by the Cauchy-Goursat theorem,

$$\int_{\gamma} \frac{(-x)^{s-1}}{e^x - 1} ds = 0$$

Since $\frac{(-x)^{s-1}}{e^x-1}$, being entire in s and holomorphic in x on C, is also continuous on C and, and since $\int_C \frac{(-x)^{s-1}}{e^x-1} dx$ is uniformly convergent in any closed disk $|s| \leq M$, we have

$$\oint_{\gamma} \int_C \frac{(-x)^{s-1}}{e^x - 1} dx ds = \int_C \oint_{\gamma} \frac{(-x)^{s-1}}{e^x - 1} ds dx$$
$$= 0$$

so that by Morera's theorem,

$$\int_C \frac{(-x)^{s-1}}{e^x - 1} dx$$

is entire in s.

Next, we consider the ray of $\mathcal{H}$ which is above the positive real axis. Letting $\log(-x) = \varrho + \varepsilon i, 0 < \varepsilon < \pi/2, s = \sigma + ti, |s| \leq M$, we have

$$\begin{aligned}\frac{e^{(\log(x)-\pi i)(s-1)}}{e^x - 1} &= \frac{e^{\varrho\sigma + (\pi-\varepsilon)t + (\varrho t + \sigma(\varepsilon - \pi))i}}{e^x - 1} \\ \left|\frac{e^{(\log(x)-\pi i)(s-1)}}{e^x - 1}\right| &\leq \frac{e^{\varrho\sigma + (\pi-\varepsilon)t}}{e^\rho - 1} \\ &\leq \frac{e^{\varrho M} e^{\pi M}}{e^\rho - 1} \\ &\leq \frac{(\rho^2 + \iota^2)^{M/2} e^{\pi M}}{e^\rho - 1}\end{aligned}$$

where ι is the constant imaginary part of x. Clearly now,

$$\int_\infty^\delta \frac{(\rho^2 + \iota^2)^{M/2} e^{\pi M}}{e^\rho - 1} d\rho$$

for some $\delta > 0$ is convergent. It follows by the Weierstrass M test that on any closed disk $|s| \leq M$,

$$\int_\infty^\delta \frac{e^{(\log(x)-\pi i)(s-1)}}{e^x - 1} d\rho$$

is absolutely and uniformly convergent. It now follows by a similar use of the Cauchy-Goursat theorem followed by Morera's theorem, that

$$\int_\infty^\delta \frac{e^{(\log(x)-\pi i)(s-1)}}{e^x - 1} d\rho$$

is an entire function of s. With minor modifications, it can be shown that

$$\int_\delta^\infty \frac{e^{(\log(x)+\pi i)(s-1)}}{e^x - 1} d\rho$$

is an entire function of s.

The proposition now follows. □

Proposition 8. *We can modify the Hankel contour $\mathcal{H}$ described above by slowly decreasing the radius ϵ of the semicircle, decreasing δ (so that the rays continue touching the semicircle), and decreasing the distance of the rays from the positive real axis, while leaving $\int_{\mathcal{H}} \frac{(-x)^{s-1}}{e^x-1} dx$ unchanged in value.*

Proof. We initially consider the rays to finite at first, extending, say, from $\Re(x) = \delta$ to $\Re(x) = X$ and denote the resulting contour by $\mathcal{H}_1$. Let $\mathcal{H}_2$ be the slightly deformed version of $\mathcal{H}_1$ as in the hypothesis. We join the two ends above the positive real axis of the rays of $\mathcal{H}_1$ and $\mathcal{H}_2$ at $\Re(x) = X$ by a small vertical segment, say $\mathcal{C}_1$. Similarly for the ends of the rays below the real axis. Denote this segment by $\mathcal{C}_2$.

Let $\mathcal{R}$ be the resulting contour, positively oriented. Then, since the integrand is holomorphic in the region bounded by $\mathcal{R}$, we have, by Cauchy's theorem,

$$\int_{\mathcal{R}} \frac{(-x)^{s-1}}{e^x - 1} dx = 0$$

Now the integrand is bounded on each of $\mathcal{C}_1$ and $\mathcal{C}_2$. Letting $X \to \infty$, while the lengths of $\mathcal{C}_1$ and $\mathcal{C}_2$ remain constant, we find by the ML-inequality that then

$$\int_{\mathcal{C}_1} \frac{(-x)^{s-1}}{e^x - 1} dx \to 0, \int_{\mathcal{C}_2} \frac{(-x)^{s-1}}{e^x - 1} dx \to 0$$

Hence, as $X \to \infty$,

$$\int_{\mathcal{H}_1} \frac{(-x)^{s-1}}{e^x - 1} dx \to \int_{\mathcal{H}_2} \frac{(-x)^{s-1}}{e^x - 1} dx$$

□

Proposition 9. *The Riemann zeta function can be extended to the whole complex plane as a meromorphic function with only a simple pole at* 1*.*

Proof. We suppose that $\Re(s) > 1$. Then, as in the previous proposition,

$$\int_{\mathcal{H}} \frac{(-x)^{s-1}}{e^x - 1} dx = \int_{\infty}^{\delta} \frac{e^{(\log(x) - \pi i)(s-1)}}{e^x - 1} d\rho + \int_{C} \frac{(-x)^{s-1}}{e^x - 1} dx + \int_{\delta}^{\infty} \frac{e^{(\log(x) + \pi i)(s-1)}}{e^x - 1} d\rho$$

Now, we will deform $\mathcal{H}$ by letting $\delta \to 0+$ and letting C, the circular arc of $\mathcal{H}$, shrink into the origin as described above, noting that the value of the integral is not changed. We see firstly that when we do so

$$\int_{C} \frac{(-x)^{s-1}}{e^x - 1} dx \to 0$$

For, if $\Re(s) = \sigma > 1$,

$$\begin{aligned} \left| \int_{C} \frac{(-x)^{s-1}}{e^x - 1} dx \right| &\leq \int_{C} \left| \frac{(-x)^{s-1}}{e^x - 1} \right| |dx| \\ &\leq \int_{C} \left| \frac{(-x)^{\sigma-1}}{e^x - 1} \right| |dx| \end{aligned}$$

But now,

$$\frac{1}{e^x - 1}$$

is holomorphic on C except for a simple pole at $x = 0$ so that

$$\frac{(-x)^{\sigma-1}}{e^x - 1}$$

is holomorphic on C because $\sigma > 1$. Hence,

$$\frac{(-x)^{\sigma-1}}{e^x - 1}$$

is bounded on C; so that, if the length of C is denoted by L, we have (ML-inequality)

$$\left|\int_C \frac{(-x)^{s-1}}{e^x - 1} dx\right| \leq ML$$

where M is some positive number. Hence, as C shrinks onto the origin, we have $L \to 0$ and so

$$\left|\int_C \frac{(-x)^{s-1}}{e^x - 1} dx\right| \to 0$$

Hence, we have in the limit as $\delta \to 0+$ and the rays fall onto the positive real axis,

$$\begin{aligned}
\int_{\mathcal{H}} \frac{(-x)^{s-1}}{e^x - 1} dx &= \int_\infty^0 \frac{e^{(\log(x) - \pi i)(s-1)}}{e^x - 1} dx + \int_0^\infty \frac{e^{(\log(x) + \pi i)(s-1)}}{e^x - 1} dx \\
&= \int_0^\infty \frac{e^{(\log(x) + \pi i)(s-1)}}{e^x - 1} dx - \int_0^\infty \frac{e^{(\log(x) - \pi i)(s-1)}}{e^x - 1} dx \\
&= \left(e^{\pi i(s-1)} - e^{-\pi i(s-1)}\right) \int_0^\infty \frac{e^{(s-1)\log x}}{e^x - 1} dx \\
&= 2i \sin(\pi(s-1)) \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \\
&= -2i \sin(\pi s) \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx
\end{aligned}$$

When $\Re(s) > 1$, this becomes

$$\int_{\mathcal{H}} \frac{(-x)^{s-1}}{e^x - 1} dx = -2i \sin(\pi s) \Gamma(s) \zeta(s)$$

The left side being entire and the right side being holomorphic for $\Re(s) > 1$, it follows that the left-side gives the unique analytic continuation of the left-hand side over the entire complex s-plane. So that, for $s \in \mathbb{C}$, we have

$$-2i \sin \pi s \Gamma(s) \zeta(s) = \int_{\mathcal{H}} \frac{(-x)^{s-1}}{e^x - 1} dx$$

By using Euler's reflection formula for the gamma function, we have

$$\zeta(s) = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathcal{H}} \frac{(-x)^{s-1}}{e^x - 1} dx$$

When $s = 1$, we have

$$\begin{aligned}
\int_{\mathcal{H}} \frac{1}{e^x - 1} dx &= \int_{\mathcal{H}} \frac{1}{x} - \frac{1}{2} + \frac{x}{12} + \ldots dx \\
&= 2\pi i
\end{aligned}$$

by the residue theorem. This implies that $\zeta(s)$ has a simple pole at $s = 1$. When $s = 2, 3, \dots$, the integral is easily seen to be zero by Cauchy's theorem, and this zero cancels the simple pole of $\Gamma(1-s)$ at those values of s. Also , when s is equal to a negative even integer, we can see by the residue theorem that ζ is then equal to zero. □

Proposition 10. *We prove the functional equation for the Riemann zeta function*

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s)$$

Proof. Let us truncate the Hankel contour $\mathcal{H}$ considered above at some value positive value $\Re(x) = M$ and call the resulting contour, $\mathcal{H}_1$; then, with center at the origin, we make an arc of a circle of radius M which joins the two open ends of $\mathcal{H}_1$; the latter arc we call $\mathcal{C}$; the whole closed curve, oriented positively, we call C, i.e.

$$C = \mathcal{C} - \mathcal{H}_1$$

We now consider the integral

$$\oint_C \frac{(-x)^{s-1}}{e^x - 1}dx$$

when $\Re(s) = \sigma < 0$. Let $n \in \mathbb{Z}\backslash\{0\}$. Then

$$\lim_{x\to 2n\pi i}\frac{x - 2n\pi i}{e^x - 1} = \lim_{r\to 2n\pi i}\frac{1}{e^x} = -1$$

Hence, the integrand has a simple pole at $x = 2n\pi i, n \in \mathbb{Z}\backslash\{0\}$. Now,

$$\left|\int_{\mathcal{C}}\frac{(-x)^{s-1}}{e^x - 1}dx\right| \leq \frac{2\pi M^\sigma}{e^M - 1} \to 0, M \to 0$$

Hence, by the residue theorem,

$$-\int_{\mathcal{H}}\frac{(-x)^{s-1}}{e^x - 1}dx = (2\pi i)\sum_{n=-\infty}^{\infty}(-n2\pi i)^{s-1}, \Re(s) < 0$$

$$\begin{aligned}\therefore \int_{\mathcal{H}}\frac{(-x)^{s-1}}{e^x - 1}dx &= -i(2\pi)^s\left((-i)^{s-1} + i^{s-1}\right)\sum_{n=1}^{\infty}n^{1-s}\\ &= -i(2\pi)^s\left((-i)^{s-1} + i^{s-1}\right)\zeta(1-s)\end{aligned}$$

Since on the said domain both sides are holomorphic, we therefore have

$$\begin{aligned}
-2i\sin(\pi s)\Gamma(s)\zeta(s) &= -i(2\pi)^s\left((-i)^{s-1}+i^{s-1}\right)\zeta(1-s)\\
2\sin(\pi s)\Gamma(s)\zeta(s) &= (2\pi)^s\left((i^{-(s-1)})+i^{s-1}\right)\zeta(1-s)\\
&= (2\pi)^s\left((-1)^{-\frac{s-1}{2}}+(-1)^{\frac{s-1}{2}}\right)\zeta(1-s)\\
&= (2\pi)^s\left(e^{-i\pi\left(\frac{s-1}{2}\right)}+e^{i\pi\left(\frac{s-1}{2}\right)}\right)\zeta(1-s)\\
&= 2(2\pi)^s\cos\left\{\pi\left(\frac{s-1}{2}\right)\right\}\zeta(1-s)\\
&= 2(2\pi)^s\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)\\
\sin(\pi s)\Gamma(s)\zeta(s) &= (2\pi)^s\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)
\end{aligned}$$

On making use of Legendre's duplication formula followed by Euler's reflection formula for the gamma function we have

$$\begin{aligned}
\sin(\pi s)(2\pi)^{-\frac{1}{2}}2^{s-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s) &= (2\pi)^s\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)\\
\sin(\pi s)(2\pi)^{-\frac{1}{2}}2^{s-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\frac{\pi}{\Gamma\left(\frac{1-s}{2}\right)\sin\left(\pi\left\{\frac{s+1}{2}\right\}\right)}\zeta(s) &= (2\pi)^s\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\sin(\pi s)}{2\cos\left(\frac{\pi s}{2}\right)}\pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{s-1}\Gamma\left(\frac{1-s}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)\\
\pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{s-1}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\\
\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)
\end{aligned}$$

□

Proposition 11.

$$\int_{-\infty}^{\infty} e^{-y^2\pi x}e^{-2n\pi iy}dy = x^{-\frac{1}{2}}e^{-\frac{n^2\pi}{x}}$$

Proof. Now

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-y^2}\cos(2ny)dy &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-y^2+2nyi}dy + \frac{1}{2}\int_{-\infty}^{\infty} e^{-y^2-2nyi}dy\\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-(y-ni)^2-n^2}dy + \frac{1}{2}\int_{-\infty}^{\infty} e^{-(y+ni)^2-n^2}dy\\
&= \frac{1}{2}\int_{-\infty+ni}^{\infty+ni} e^{-s^2-n^2}ds + \frac{1}{2}\int_{-\infty-ni}^{\infty-ni} e^{-s^2-n^2}ds
\end{aligned}$$

where, in each integral, we have substituted for the real variable y a complex variable s, and the real integrals become contour integrals over the lines $s = ni$ and $s = -ni$ respectively. Consider the first integral(the case for the second integral is similar). If we denote by R the rectangle with vertices at

$$(-M, 0), (M, 0), (M, ni), (-M, ni), M > 0$$

in the complex s-plane, then by Cauchy's theorem we have that

$$\oint_R e^{-s^2-n^2} ds = 0$$

Letting not $M \to 0$, we easily see, by the ML- inequality, for example, that the integrals over the vertical sides of R vanish. Hence, we end up with

$$\int_{-\infty}^{\infty} e^{-y^2} \cos(2ny) dy = \frac{1}{2} \int_{-\infty}^{\infty} e^{-y^2-n^2} dy + \frac{1}{2} \int_{-\infty}^{\infty} e^{-y^2-n^2} dy = \sqrt{\pi} e^{-n^2}$$

Next, we easily see that

$$\int_{-\infty}^{\infty} e^{-y^2} \sin(2ny) dy = 0$$

because the integrand is an odd function of y. It now follows that

$$\int_{-\infty}^{\infty} e^{-y^2} e^{-2nyi} dy = \sqrt{\pi} e^{-n^2}$$

The substitution $y \to y\sqrt{\pi x}$ followed by $b \to \sqrt{\frac{\pi}{x}}$ now give the required result. □

Proposition 12. *Let*

$$\psi(x) = \sum_{t=1}^{\infty} e^{-t^2 \pi x}, x > 0$$

Then

$$\psi(x) + 1 = x^{-\frac{1}{2}} \left(2\psi\left(\frac{1}{x}\right) + 1 \right)$$

Proof. Consider the infinite series

$$\sum_{t=-\infty}^{\infty} e^{-t^2 \pi x} = 2\psi(x) + 1$$

It is easily seen, by the comparison test with $\sum_{t=1}^{\infty} e^{-t\pi x}$, for example, that $\psi(x)$ is absolutely convergent. Hence, rearrangement of terms does not affect the value of

$$\sum_{-\infty}^{\infty} e^{-t^2 \pi x}$$

It follows that if $\beta \in \mathbb{Z}$, then

$$\sum_{t=-\infty}^{\infty} e^{-t^2 \pi x} = \sum_{t=-\infty}^{\infty} e^{-(t+\beta)^2 \pi x}$$

so that if we let

$$f(z) = \sum_{t=-\infty}^{\infty} e^{-(t+z)^2 \pi x}, x > 0$$

it follows that $f(z)$ is a periodic function (of the continuous real variable z) with period 1. Moreover, it is easy to see, by the Weierstrass M test, for example, that $\psi(x)$ and the corresponding termwise-differentiated series are uniformly convergent on any interval $x \geq \epsilon, \epsilon > 0$. Thus it can be seen that $f(z)$ satisfies the Dirichlet conditions for Fourier series are satisfied. We therefore have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi i z}$$

where

$$c_n = \int_{1/2}^{1/2} f(y) e^{-2n\pi i y} dy = \int_0^1 f(y) e^{-2n\pi i y} dy$$

We therefore have

$$\begin{aligned} c_n &= \int_0^1 \sum_{t=-\infty}^{\infty} e^{-(t+y)^2 \pi x} e^{-2n\pi i y} dy \\ &= \sum_{t=-\infty}^{\infty} \int_0^1 e^{-(t+y)^2 \pi x} e^{-2n\pi i y} dy \\ &= \int_{-\infty}^{\infty} e^{-y^2 \pi x} e^{-2n\pi i y} dy \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{t=-\infty}^{\infty} e^{-t^2 \pi x} &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y^2 \pi x} e^{-2n\pi i y} dy \\ &= x^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2 \pi}{x}} \end{aligned}$$

from which the required result easily follows using the previous proposition. □

Proposition 13. *With $\psi(x)$ defined as in the previous proposition, we define next*

$$\xi(s) = \frac{1}{2} s(s-1) \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s), s \in \mathbb{C}$$

(we note that $\xi(s)$ is an entire function which satisfies the functional equation $\xi(s) = \xi(1-s)$) and we also define the Riemann Xi function,

$$\Xi(t) = \xi\left(\frac{1}{2} + ti\right), t \in \mathbb{C}$$

Then

$$\Xi(t) = 4\int_1^\infty \frac{d\left(x^{\frac{3}{2}}\psi'(x)\right)}{dx} x^{-\frac{1}{4}} \cos\left(\frac{1}{2} t \log x\right) dx, t \in \mathbb{C}$$

Proof.

$$\begin{aligned}
\Gamma(s) &= \int_0^\infty e^{-x} x^{s-1} dx, \Re(s > 0) \\
\Gamma\left(\frac{s}{2}\right) &= \int_0^\infty e^{-x} x^{\frac{s}{2}-1} dx \\
\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} &= \int_0^\infty e^{-x\pi} x^{\frac{s}{2}-1} dx \\
\frac{1}{n^s}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} &= \int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1} dx \\
\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) &= \sum_{n=1}^\infty \int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1} dx, \Re(s) > 1
\end{aligned}$$

The summation and integration signs can be interchanged, the justification having already been given in a previous proposition,

$$\begin{aligned}
\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) &= \int_0^\infty \psi(x) x^{\frac{s}{2}-1} dx, \Re(s) > 1 \\
&= \int_1^\infty \psi(x) x^{\frac{s}{2}-1} dx + \int_0^1 \psi(x) x^{\frac{s}{2}-1} dx
\end{aligned}$$

In the second integral, we make use of the result in the previous proposition to obtain

$$\begin{aligned}
\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) &= \int_1^\infty \psi(x) x^{\frac{s}{2}-1} dx + \int_0^1 \psi\left(\frac{1}{x}\right) x^{\frac{s-3}{2}} dx + \frac{1}{2}\int_0^1 \left(x^{\frac{s-3}{2}} - x^{\frac{s}{2}}\right) dx \\
&= \int_1^\infty \psi(x) x^{\frac{s}{2}-1} dx + \int_0^1 \psi\left(\frac{1}{x}\right) x^{\frac{s-3}{2}} dx + \frac{1}{s(s-1)}
\end{aligned}$$

In the second integral, we make the substitution $x \to \frac{1}{x}$ to obtain

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \psi(x)(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}) dx, \Re(s) > 1$$

Now, by the Weierstrass M test, it is readily seen that $\psi(x)$ is uniformly convergent on the interval $x \geq 1$. So that,

$$\lim_{x\to\infty}(x^s\psi(x)) = \sum_{n=1}^\infty \lim_{x\to\infty}(x^s e^{-n^2\pi x})$$

now by l'Hôpital's rules it is easily seen that

$$\lim_{x\to\infty} |x^s e^{-n^2\pi x}| = 0, s \in \mathbb{C}$$

and it follows that

$$\int_1^\infty \psi(x) x^s dx$$

is absolutely and, by the Weierstrass M test, for example, uniformly convergent in any closed disk $|s| < R, R > 0$. As in previous propositions, we can make use of the Cauchy-Goursat and Morera theorems and deduce that $\int_1^\infty \psi(x)x^s dx$ is an entire function. So that we have

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \psi(x)(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}})dx, s \in \mathbb{C}$$

Therefore

$$\xi(s) = \frac{1}{2} + \frac{1}{2}s(s-1)\int_1^\infty \psi(x)(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}})dx, s \in \mathbb{C}$$

(We note easily that this supplies a second proof of the functional equation)

$$\xi(s) = \frac{1}{2} + \frac{1}{2}s(s-1)\int_1^\infty \frac{d}{dx}\left\{\psi(x)\left(\frac{2}{s}x^{\frac{s}{2}} + \frac{2}{1-s}x^{\frac{1-s}{2}}\right)\right\}dx$$
$$- \frac{1}{2}s(s-1)\int_1^\infty \psi'(x)\left(\frac{2}{s}x^{\frac{s}{2}} + \frac{2}{1-s}x^{\frac{1-s}{2}}\right)dx$$

By the ratio test we easily prove that $\sum_{n=1}^\infty n^2 e^{-n^2}$ is convergent. Hence $-\pi\sum_{n=1}^\infty n^2 e^{-n^2\pi x}$ is uniformly convergent on the interval $x \geq 1$; so that $\psi'(x) = -\pi\sum_{n=1}^\infty n^2 e^{-n^2\pi x}$ and $\psi'(x)$ is continuous for $x \geq 1$. We may thus apply the Fundamental theorem of Calculus in the first integral, to obtain

$$\xi(s) = \frac{1}{2} + \psi(1) + \int_1^\infty \psi'(x)\left[(1-s)x^{\frac{s}{2}} + sx^{\frac{1-s}{2}}\right]dx$$
$$= \frac{1}{2} + \psi(1) + \int_1^\infty x^{\frac{3}{2}}\psi'(x)\left[(1-s)x^{\frac{s-1}{2}-1} + sx^{-\frac{s}{2}-1}\right]dx$$

This gives

$$\xi(s) = \frac{1}{2} + \psi(1) + \int_1^\infty \frac{d}{dx}\left(x^{\frac{3}{2}}\psi'(x)\left(-2x^{\frac{s-1}{2}} - 2x^{-\frac{s}{2}}\right)\right)dx$$
$$- \int_1^\infty \frac{d\left\{x^{\frac{3}{2}}\psi'(x)\right\}}{dx}\left(-2x^{\frac{s-1}{2}} - 2x^{-\frac{s}{2}}\right)dx$$

We can now, similarly as before, justify a second use of the Fundamental theorem of Calculus in the first integral and this gives

$$\xi(s) = \frac{1}{2} + \psi(1) - 4\psi'(1) + \int_1^\infty \frac{d\left\{x^{\frac{3}{2}}\psi'(x)\right\}}{dx}\left(2x^{\frac{s-1}{2}} + 2x^{-\frac{s}{2}}\right)dx$$

We may now deduce, by differentiating both sides of the equation in the result of the last proposition, that the constant term is equal to 0; so that

$$\xi(s) = 2\int_1^\infty \frac{d\left\{x^{\frac{3}{2}}\psi'(x)\right\}}{dx}\left(x^{\frac{s-1}{2}} + x^{-\frac{s}{2}}\right)dx$$

giving

$$\xi(s) = 4\int_1^\infty \frac{d\left\{x^{\frac{3}{2}}\psi'(x)\right\}}{dx}x^{-\frac{1}{4}}\cosh\left(\frac{1}{2}\left(s-\frac{1}{2}\right)\log x\right)dx, s \in \mathbb{C}$$

Finally, we can put $s = \frac{1}{2} + ti, t \in \mathbb{C}$ to get

$$\Xi(t) = 4\int_1^\infty \frac{d\left(x^{\frac{3}{2}}\psi'(x)\right)}{dx}x^{-\frac{1}{4}}\cos\left(\frac{1}{2}t\log x\right)dx, t \in \mathbb{C}$$

□

1.2 Zeros and Product

Proposition 14. *The roots of* $\Xi(t) = 0$ *lie in the strip* $-\frac{1}{2} \leq \Im t \leq \frac{1}{2}$.

Proof. It is easily seen that ξ satisfies the functional equation $\xi(s) = \xi(1-s)$ so that $\Xi(t) = \Xi(-t)$. Now since

$$\zeta(s) = \prod_{p \text{ prime}} (1-p^{-s})^{-1}, \Re s > 1$$

Since none of the factors on the right are able to vanish, we conclude that ζ has no zeros on the open half-plane $\Re(s) > 1$. Since the same holds true for the other factors of $\xi(s)$, it follows that the roots of $\xi(s) = 0$ lie in the strip $0 \leq \Re(s) \leq 1$. Equivalently, Ξ can vanish only if $-\frac{1}{2} \leq \Im t \leq \frac{1}{2}$ □

Proposition 15. *The zeros of* ξ *are symmetrical with respect to both of the lines* $\Re(s) = \frac{1}{2}$ *and* $\Im(s) = 0$.

Proof. That the zeros are symmetrical with respect to the line $\Re(s) = \frac{1}{2}$ follows from the functional equation $\xi(s) = \xi(1-s)$.

Next, we will show that $\xi(s) = \overline{\xi(\overline{s})}$ where for a complex number z, its complex conjugate is denoted by $\overline{z}$.

Firstly, we show that $\overline{\xi(\overline{s})}$ is an entire function. For since $\xi(s)$ is holomorphic, we have that

$$\lim_{\Delta s \to 0} \frac{\xi(s+\Delta s) - \xi(s)}{\Delta s}$$

exists independent of the manner in which $\Delta s \to 0$ and it is equal to $\xi'(s)$. We therefore have

$$\xi(s) = \lim_{\Delta s \to 0} \frac{\xi(s+\overline{\Delta s}) - \xi(s)}{\overline{\Delta s}}$$

But now,we have that

$$\begin{aligned}
\lim_{\Delta s\to 0}\frac{\left(\overline{\xi(\overline{s+\Delta s})}-\overline{\xi(\overline{s})}\right)}{\Delta s} &= \lim_{\Delta s\to 0}\frac{\left(\overline{\xi(\overline{s}-\overline{\Delta s})-\xi(\overline{s})}\right)}{\Delta s}\\
&= \lim_{\Delta s\to 0}\frac{\left(\overline{\xi(\overline{s}-\overline{\Delta s})-\xi(\overline{s})}\right)\overline{\Delta s}}{\Delta s\overline{\Delta s}}\\
&= \overline{\lim_{\Delta s\to 0}\frac{\left(\xi(\overline{s}-\overline{\Delta s})-\xi(\overline{s})\right)\Delta s}{\Delta s\overline{\Delta s}}}\\
&= \overline{\lim_{\Delta s\to 0}\frac{\left(\xi(\overline{s}-\overline{\Delta s})-\xi(\overline{s})\right)}{\overline{\Delta s}}}\\
&= \overline{\xi'(\overline{s})}
\end{aligned}$$

The right-side is well defined because ξ is an entire function and this means that the derivative

$$\lim_{\Delta s\to 0}\frac{\left(\overline{\xi(\overline{s+\Delta s})}-\overline{\xi(\overline{s})}\right)}{\Delta s}$$

is well-defined on the whole complex s-plane and so

$$\overline{\xi(\overline{s})}$$

is an entire function.

Now it is not difficult to see, from the definition of ξ, that $\xi(s)$ is real on the real axis; for firstly, γ, exception made for the poles, is real on the real axis; secondly, ζ being obviously real for real numbers larger than 1, we see that ζ is also real for real numbers less than 0 by making use of its functional equation; as for the interval $[0,1]$, we, to begin, disregard the pole at 1 (and so we also disregard the value at 0, which, by the way, is not difficult to evaluate as $\zeta(0)=-\frac{1}{2}$) and then we recall the representation of ζ in terms of the Dirichlet eta function (which is also, in passing, easy to show to be valid on the entire complex plane) to conclude that ζ is real on $(0,1)$. We conclude that ξ is real on the real axis.

Now, we see that the two entire functions $\xi(s)$ and $\overline{\xi(\overline{s})}$ agree on the real axis; we conclude therefore that they agree, by analytic continuation, on the entire complex plane, i.e.

$$\xi(s)=\overline{\xi(\overline{s})}, s\in\mathbb{C}$$

Finally, the last equation shows that if ρ is a zero of the function ξ, then so is $\overline{\rho}$ and since moreover we have shown that $1-\rho$ is also a root, it follows that the roots of the function ξ, which are the non-trivial zeros of the Riemann zeta functions, exhibit a symmetry with respect to the two lines $\Re(s)=\frac{1}{2}$ and $\Im(s)=0$.

Equivalently, the zeros of Ξ are symmetrical with respect to both the real and imaginary axes; i.e. if t is a zero of Ξ, then so are $-t,\overline{t}$, and $-\overline{t}$. □

Proposition 16. *The ξ function has no zeros on the real axis.*

Proof. It has already been shown previously that ξ has no zeros for $\Re(s) < 0$ and $\Re(s) > 1$. We therefore only remain to show this for $s \in \mathbb{R}, 0 \leq s \leq 1$.

Firstly, consider $0 < s < 1$. Since

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

we easily see that none of the factors $s, s-1, \pi^{-\frac{s}{2}}$ vanish for $0 < s < 1$. We have also shown that the gamma function has no zeros. Moreover, from the representation of $\zeta(s)$ in terms of the Dirichlet eta function for $\Re(s) > 0$, we see that $\zeta(s)$ has no zeros on the real segment $0 < s < 1$. Hence, $\xi(s)$ does not vanish for $0 < s < 1$.

Next have

$$\xi(0) = -\frac{1}{2}\zeta(0)\lim_{s\to 0}\left\{s\Gamma\left(\frac{s}{2}\right)\right\}$$

We calculate $\zeta(0)$ by remembering that we have

$$\zeta(s) = \frac{i\Gamma(1-s)}{2\pi}\int_{\mathcal{H}}\frac{(-x)^{s-1}}{e^x-1}dx$$

where $\mathcal{H}$ denotes a Hankel contour from $+\infty$ to $+\infty$ and does enclose as point of discontinuity only 0, so that

$$\zeta(0) = \frac{1}{2\pi i}\int_{\mathcal{H}}\frac{dx}{x(e^x-1)}$$

Now

$$\frac{1}{x(e^x-1)} = \frac{1}{x^2} - \frac{1}{2x} + \frac{1}{12} + \dots$$

so that, by the residue theorem,

$$\int_{\mathcal{H}}\frac{dx}{x(e^x-1)} = -\frac{1}{2}(2\pi i)$$

Hence,

$$\zeta(0) = -\frac{1}{2}$$

This gives

$$\xi(0) = \frac{1}{4}\lim_{s\to 0}\left\{s\Gamma\left(\frac{s}{2}\right)\right\}$$

Hence, using the Euler reflection formula for the gamma function,

$$\begin{aligned}\xi(0) &= \frac{1}{4}\lim_{s\to 0}\left\{s\frac{\pi}{\Gamma\left(1-\frac{s}{2}\right)\sin\left(\frac{\pi s}{2}\right)}\right\}\\ &= \frac{1}{2}\end{aligned}$$

It follows finally from the functional equation for the ξ function that

$$\xi(1) = \xi(0) = \frac{1}{2} \neq 0$$

□

Proposition 17. *For $\Re(s) > 0$, we have*

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{|s|}\right)$$

Lemma 17.1. *Define*

$$f(s) = se^{\gamma s}\prod_{n=1}^{\infty}\left\{\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}\right\}, s \in \mathbb{C}$$

where γ is the Euler-Mascheroni constant. Then $f(s)$ is an entire function.

Proof. We will show that $f(s)$ is holomorphic in any closed disk of arbitrary radius.

If, for a positive integer N, we have $|s| \leq \frac{1}{2}N$ and $n > N$, then

$$\begin{aligned}\log\left(1+\frac{s}{n}\right) - \frac{s}{n} &= \left|\sum_{m=2}^{\infty}(-1)^{m-1}\frac{s^m}{mn^m}\right| \\ &\leq \frac{|s|^2}{n^2}\sum_{m=0}^{\infty}\left|\frac{s}{n}\right|^m \\ &\leq \frac{N^2}{4n^2}\sum_{m=0}^{\infty}\frac{1}{2^m} \\ &= \frac{N^2}{2n^2}\end{aligned}$$

It therefore follows by the Weierstrass M test that

$$\sum_{n=N+1}^{\infty}\left\{\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right\}$$

is absolutely and uniformly convergent in the closed disk $|s| \leq \frac{1}{2}N$. Let C be a simple closed curve lying in that disk. Since the summands are holomorphic, therefore

$$\sum_{n=N+1}^{k}\left\{\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right\}, k \in \mathbb{N}$$

is holomorphic in the disk and so, by the Cauchy-Goursat theorem, we have

$$\oint_C \sum_{n=N+1}^{k}\left\{\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right\} = 0$$

By the absolute and uniform convergence of the infinite series, we have

$$\begin{aligned}\lim_{k\to\infty}\oint_C \sum_{n=N+1}^{k}\left\{\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right\} &= \oint_C \lim_{k\to\infty}\sum_{n=N+1}^{k}\left\{\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right\} \\ &= \oint_C \sum_{n=N+1}^{\infty}\left\{\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right\}\end{aligned}$$

so that by Morera's theorem, it follows that

$$\sum_{n=N+1}^{\infty} \left\{ \log\left(1+\frac{s}{n}\right) - \frac{s}{n} \right\}$$

is an entire function; hence $f(s)$ is also entire. □

Lemma 17.2. *With $f(s)$ defined as in the previous lemma, we have*

$$\{f(s)\}^{-1} = \frac{1}{s}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1}\right\}, s \in \mathbb{C}/\{0,-1,-2,\dots\}$$

Proof. Using the previous lemma,

$$\begin{aligned}
f(s) &= s \lim_{m\to\infty}\left[e^{\left(\sum_{i=1}^{m}\frac{1}{i} - \log m\right)} \prod_{n=1}^{m}\left\{\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}\right\}\right] \\
&= s \lim_{m\to\infty}\left[m^{-s}\prod_{n=1}^{m}\left(1+\frac{s}{n}\right)\right] \\
&= s \lim_{m\to\infty}\left[\prod_{n=1}^{m-1}\left(1+\frac{1}{n}\right)^{-s}\prod_{n=1}^{m}\left(1+\frac{s}{n}\right)\right] \\
&= s \lim_{m\to\infty}\left[\prod_{n=1}^{m}\left\{\left(1+\frac{s}{n}\right)\left(1+\frac{1}{n}\right)^{-s}\right\}\left(1+\frac{1}{m}\right)^{s}\right]
\end{aligned}$$

from which the lemma follows. □

Lemma 17.3. *For $f(s)$ defined as in the previous lemmas, we have the functional equation*

$$\frac{1}{f(s+1)} = s\frac{1}{f(s)}$$

Proof. From the previous lemma,

$$\begin{aligned}
\frac{f(s)}{f(s+1)} &= \frac{1}{s+1}\left[\lim_{m\to\infty}\prod_{n=1}^{m}\frac{\left(1+\frac{1}{n}\right)^{s+1}}{1+\frac{s+1}{n}}\right]\left[\frac{1}{s}\lim_{m\to\infty}\prod_{n=1}^{m}\frac{\left(1+\frac{s}{n}\right)^{s}}{1+\frac{s}{n}}\right]^{-1} \\
&= \frac{s}{s+1}\lim_{m\to\infty}\prod_{n=1}^{m}\left\{\frac{\left(1+\frac{1}{n}\right)(s+n)}{s+n+1}\right\} \\
&= s\lim_{m\to\infty}\frac{m+1}{s+m+1} \\
&= s
\end{aligned}$$

□

Lemma 17.4. *We have the inequality*

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq n^{-1}t^2e^{-t}; t, n > 0, \frac{t}{n} < 1$$

Proof. Using the Maclaurin expansions, we easily deduce that

$$1 + y \leq e^y \leq (1-y)^{-1}, 0 \leq y < 1$$

Putting $y = \frac{t}{n}$, we obtain

$$\left(1 - \frac{t}{n}\right)^n \leq e^{-t} \leq \left(1 + \frac{t}{n}\right)^{-n}$$

We therefore have

$$\begin{aligned} 0 &\leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \\ &= e^{-t}\left\{1 - e^t\left(1 - \frac{t}{n}\right)^n\right\} \\ &\leq e^{-t}\left\{1 - \left(1 - \frac{t^2}{n^2}\right)^n\right\} \end{aligned}$$

Now from Bernoulli's inequality it follows that we have, for $0 \leq \alpha \leq 1, n \geq 1$,

$$(1-\alpha)^n \geq 1 - n\alpha$$

so that

$$1 - \left(1 - \frac{t^2}{n^2}\right)^n \leq \frac{t^2}{n}$$

from which the lemma now follows. □

Lemma 17.5. *For $f(s)$ defined as in the previous propositions, we have*

$$f(s) = \frac{1}{\Gamma(s)}$$

Proof. We define

$$\Pi(s,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1}dt, \Re(s) > 0$$

and make the substitution $t \to nt$ to obtain

$$\Pi(s,n) = n^s \int_0^1 (1-t)^n t^{s-1}, \Re(s) > 0$$

When n is a positive integer we may integrate by parts successively until we arrive at

$$\Pi(s,n) = \frac{n(n-1)\dots 1}{s(s+1)\dots(s+n-1)} \int_0^1 t^{s+n-1}dt = \frac{1 \cdot 2 \dots n}{s(s+1)\dots(s+n)} n^s$$

so that

$$\Pi(s,n) \to \frac{1}{f(s)}$$

as $n \to \infty$. Thus, for $\Re(s) > 0$,

$$\begin{aligned}\int_0^\infty e^{-t}t^{s-1}dt - \frac{1}{f(s)} &= \lim_{n\to\infty}\left[\int_0^n \left\{e^{-t} - \left(1-\frac{t}{n}\right)^n\right\} t^{s-1}dt + \int_n^\infty e^{-t}t^{s-1}dt\right] \\ &= \lim_{n\to\infty}\int_0^n \left\{e^{-t} - \left(1-\frac{t}{n}\right)^n\right\} t^{s-1}dt\end{aligned}$$

Using the result of the previous lemma, we find that, if we put $\Re(s) = \sigma > 0$

$$\begin{aligned}\left|\int_0^n \left\{e^{-t} - \left(1-\frac{t}{n}\right)^n\right\} t^{s-1}dt\right| &\leq \int_0^n n^{-1}e^{-t}t^{\sigma-1}dt \\ &\leq \frac{1}{n}\int_0^\infty e^{-t}t^{\sigma-1}dt\end{aligned}$$

and as the latter vanished as $n \to \infty$, we have

$$\frac{1}{f(s)} = \int_0^\infty e^{-t}t^{s-1}dt, \Re(s) > 0$$

Since we have defined the gamma function by the last integral and the functional equation

$$\Gamma(s+1) = s\Gamma(s)$$

and since we have shown that the function $f(s)$ satisfies the same functional equation, it follows that,

$$f(s) = \frac{1}{\Gamma(s)}, s \in \mathbb{C}$$

□

Lemma 17.6.

$$\gamma = \int_0^\infty \left\{\frac{1}{1-e^{-t}} - \frac{1}{t}\right\} e^{-t}dt$$

where γ is the Euler-Mascheroni constant.

Proof. It is easily proved by induction that

$$\int_0^1 \frac{1-(1-t)^n}{t}dt = \frac{1}{n}, n \in \mathbb{N}$$

Hence,

$$\begin{aligned}
\gamma &= \lim_{n\to\infty}\left[\int_0^1 \frac{1-(1-t)^n}{t}dt - \int_1^n \frac{dt}{t}\right]\\
&= \lim_{n\to\infty}\left[\int_0^n \frac{1-\left(1-\frac{t}{n}\right)^n}{t}dt - \int_1^n \frac{dt}{t}\right]\\
&= \lim_{n\to\infty}\left[\int_0^1 \frac{1-\left(1-\frac{t}{n}\right)^n}{t}dt + \int_1^n \frac{1-\left(1-\frac{t}{n}\right)^n}{t}dt - \int_1^n \frac{dt}{t}\right]\\
&= \lim_{n\to\infty}\left[\int_0^1 \frac{1-\left(1-\frac{t}{n}\right)^n}{t}dt - \int_1^n \left(1-\frac{t}{n}\right)^n \frac{dt}{t}\right]\\
&= \int_0^1 \frac{1-e^{-t}}{t}dt - \lim_{n\to\infty}\int_{1/n}^1 \left(1-\frac{1}{tn}\right)^n \frac{dt}{t}
\end{aligned}$$

From the inequality in a previous lemma, we have,

$$e^{-1/t}\left(1-\frac{1}{t^2 n}\right) \le \left(1-\frac{t}{n}\right)^n \le e^{-1/t}$$

from which it now follows that

$$\begin{aligned}
\gamma &= \int_0^1 \frac{1-e^{-t}}{t}dt - \int_0^1 \frac{e^{-1/t}}{t}dt\\
&= \int_0^1 \frac{1-e^{-t}}{t}dt - \int_1^\infty \frac{e^{-t}}{t}dt\\
&= \lim_{\epsilon\to 0+}\left\{\int_\epsilon^1 \frac{dt}{t} - \int_\epsilon^\infty \frac{e^{-t}}{t}dt\right\}
\end{aligned}$$

We note now that by l'Hôpital's rule we can easily see that

$$\lim_{\epsilon\to 0+}\int_{1-e^{-\epsilon}}^{\epsilon} \frac{t}{dt} = 0$$

so that

$$\begin{aligned}
\gamma &= \lim_{\epsilon\to 0+}\left\{\int_{1-e^{-\epsilon}}^{\epsilon} \frac{dt}{t} + \int_\epsilon^1 \frac{dt}{t} - \int_\epsilon^\infty \frac{e^{-t}}{t}dt\right\}\\
&= \lim_{\epsilon\to 0+}\left\{\int_{1-e^{-\epsilon}}^{1} \frac{dt}{t} - \int_\epsilon^\infty \frac{e^{-t}}{t}dt\right\}
\end{aligned}$$

In the first integral, we make now the substitution $t \to 1-e^{-t}$ to obtain

$$\begin{aligned}
\gamma &= \lim_{\epsilon\to 0+}\left\{\int_\epsilon^\infty \frac{e^{-t}}{1-e^{-t}}dt - \int_\epsilon^\infty \frac{e^{-t}}{t}dt\right\}\\
&= \int_0^\infty \left\{\frac{1}{1-e^{-t}} - \frac{1}{t}\right\} e^{-t}dt
\end{aligned}$$

□

Lemma 17.7.

$$\frac{d}{ds}\log\Gamma(s) = -\gamma - \frac{1}{s} + s\sum_{n=1}^{\infty}\frac{1}{n(s+n)}, s \in \mathbb{C}/\{0,-1,-2,\dots\}$$

Proof. We have seen that

$$\frac{1}{\Gamma(s+1)} = e^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}, s \in \mathbb{C}$$

We have, for s not a nonpositive integer, by taking logarithms on both sides,

$$-\log\Gamma(s+1) = \gamma s + \sum_{n=1}^{\infty}\left\{\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right\}$$

By what has been shown in a previous lemma about the uniform and absolute convergence on the right in closed disks excluding nonpositive integer points, we can easily see that termwise differentiation is permitted, so that

$$\frac{d}{ds}\log\Gamma(s+1) = -\gamma + \sum_{n=1}^{\infty}\left\{\frac{1}{n} - \frac{1}{s+n}\right\}$$

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} = -\gamma + \sum_{n=1}^{\infty}\frac{s}{n(s+n)}$$

from which the lemma follows, as

$$\frac{\Gamma'(1+s)}{\Gamma(1+s)} = \frac{\Gamma'(s)}{\Gamma(s)} + \frac{1}{s}$$

□

Lemma 17.8. *We have*

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} = \int_0^{\infty}\left\{\frac{e^{-t}}{t} - \frac{e^{-ts}}{e^t-1}\right\}dt, \Re(s) > 0$$

Proof. It follows form the last lemma that

$$\begin{aligned}
\frac{\Gamma'(s)}{\Gamma(s)} &= -\gamma - \int_0^\infty e^{-st}dt + \lim_{N\to\infty}\sum_{n=1}^{N}\int_0^\infty \left(e^{-nt} - e^{-(s+n)t}\right)dt \\
&= -\gamma - \int_0^\infty e^{-st}dt + \lim_{N\to\infty}\int_0^\infty \sum_{n=1}^{N}\left(e^{-nt} - e^{-(s+n)t}\right)dt, \Re(s) > 0 \\
&= -\gamma + \lim_{N\to\infty}\int_0^\infty \frac{e^{-t} - e^{-st} - e^{-(N+1)t} + e^{-(s+N+1)t}}{1-e^{-t}}dt \\
&= -\int_0^\infty \left\{\frac{1}{1-e^{-t}} - \frac{1}{t}\right\} e^{-t}dt \\
&+ \lim_{N\to\infty}\int_0^\infty \frac{e^{-t} - e^{-st} - e^{-(N+1)t} + e^{-(s+N+1)t}}{1-e^{-t}}dt \\
&= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right)dt - \lim_{N\to\infty}\int_0^\infty \frac{1-e^{-st}}{1-e^{-t}}e^{-(N+1)t}dt
\end{aligned}$$

In the left integral, the integrand is continuous for $t > 0, \Re(s) > 0$. When $0 \le t \le 1$, the integrand is bounded and has a finite limit as $t \to 0+$ by l'Hôpital's rule. Hence,

$$\int_0^1 \frac{1-e^{-st}}{1-e^{-t}}e^{-(N+1)t}dt$$

is convergent. When $t \ge 1$,

$$\left|\frac{1-e^{-st}}{1-e^{-t}}\right| |e^{-(N+1)t}| < \frac{1+e^{-st}}{1+e^{-1}}e^{-(N+1)t} \le \frac{2}{1+e^{-1}}e^{-(N+1)t}, \Re(s) > 0$$

so that

$$\int_0^\infty \frac{1-e^{-st}}{1+e^{-t}}e^{-(N+1)t}$$

is uniformly and absolutely convergent by the Weierstrass M test. We therefore have

$$\lim_{N\to\infty}\int_0^\infty \frac{1-e^{-st}}{1-e^{-t}}e^{-(N+1)t} = \int_0^\infty \lim_{N\to\infty}\frac{1-e^{-st}}{1-e^{-t}}e^{-(N+1)t} = 0$$

so that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right)dt, \Re(s) > 0$$

from which the lemma follows. □

Lemma 17.9. *We have*

$$\int_0^\infty \frac{e^{-t} - e^{-ts}}{t}dt = \log s, \Re(s) > 0$$

Proof. Let ϵ and R be positive real numbers; let s be a complex number such that $\Re(s) > 0$. Consider the quadrilateral, call it Q, with vertices at the points $\epsilon, R, \epsilon s, Rs$. Then $\frac{e^{-z}}{z}$ is holomorphic inside Q so that

$$\oint_Q \frac{e^{-z}}{z} dz = 0$$

$$\therefore \int_\epsilon^R \frac{e^{-z}}{z} dz - \int_{\epsilon s}^{Rs} \frac{e^{-z}}{z} dz = \int_\epsilon^{\epsilon s} \frac{e^{-z}}{z} dz - \int_R^{Rs} \frac{e^{-z}}{z} dz$$

Hence, for $\Re(s) > 0$,

$$\begin{aligned}\int_0^\infty \frac{e^{-t} - e^{-ts}}{t} dt &= \lim_{\substack{\epsilon \to 0+ \\ R \to \infty}} \left\{ \int_\epsilon^R \frac{e^{-z}}{z} dz - \int_\epsilon^R \frac{e^{-sz}}{z} dz \right\} \\ &= \lim_{\substack{\epsilon \to 0+ \\ R \to \infty}} \left\{ \int_\epsilon^R \frac{e^{-z}}{z} dz - \int_{\epsilon s}^{Rs} \frac{e^{-z}}{z} dz \right\} \\ &= \lim_{\substack{\epsilon \to 0+ \\ R \to \infty}} \left\{ \int_\epsilon^{\epsilon s} \frac{e^{-z}}{z} dz - \int_R^{Rs} \frac{e^{-z}}{z} dz \right\}\end{aligned}$$

We see easily by the ML inequality that the second integral tends to zero. Thus

$$\begin{aligned}\int_0^\infty \frac{e^{-t} - e^{-ts}}{t} dt &= \lim_{\epsilon \to 0+} \int_\epsilon^{\epsilon s} \frac{e^{-z}}{z} dz \\ &= \lim_{\epsilon \to 0+} \left\{ \int_\epsilon^{\epsilon s} \frac{dz}{z} - \int_\epsilon^{\epsilon s} \left(\frac{1}{z} - \frac{e^{-z}}{z} \right) dz \right\} \\ &= \log s - \lim_{\epsilon \to 0+} \int_\epsilon^{\epsilon s} \left(\frac{1}{z} - \frac{e^{-z}}{z} \right) dz\end{aligned}$$

The last integrand is easily seen to approach 1 as $z \to 0$, and the last limit is thus 0, giving the required lemma. □

Lemma 17.10. *We have*

$$\log \Gamma(s) = \left(s - \frac{1}{2} \right) \log s - s + \frac{1}{2} \log 2\pi + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-ts}}{t} dt, \Re(s) > 0$$

Proof. We have, by combining the previous two lemmas, for $\Re(s) > 0$,

$$\begin{aligned}\frac{d}{ds} \log \Gamma(1+s) &= \int_0^\infty \left\{ \frac{e^{-t}}{t} - \frac{e^{-ts}}{e^t - 1} \right\} dt \\ &= \frac{1}{2} \int_0^\infty e^{-ts} dt + \int_0^\infty \frac{e^{-t} - e^{-ts}}{t} dt - \int_0^\infty \left\{ \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right\} e^{-ts} dt \\ &= \frac{1}{2s} + \log s - \int_0^\infty \left\{ \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right\} e^{-ts} dt\end{aligned}$$

Now the last integrand is continuous for $x > 0, \Re(s) > 0$ and the integral is uniformly convergent for $\Re(s) \geq \epsilon, \epsilon > 0$, so that integration under the integral sign with respect to s from 1 to s is permitted and thus we have

$$\log \Gamma(1+s) = \left(s + \frac{1}{2}\right) \log s - s + 1 + \int_0^\infty \left\{\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right\} \frac{e^{-ts} - e^{-t}}{t} dt$$

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + 1 + \int_0^\infty \left\{\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right\} \frac{e^{-ts}}{t} dt - \int_0^\infty \left\{\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right\} \frac{e^{-t}}{t} dt$$

In the last equation, we let $s = \frac{1}{2}$ to obtain

$$\begin{aligned}
\log \sqrt{\pi} &= \frac{1}{2} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{1}{2}t}}{t} dt - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-t}}{t} dt \\
&= \frac{1}{2} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{1}{2}t}}{t} dt - \int_0^\infty \left(\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{\frac{t}{2}} - 1}\right) \frac{e^{-\frac{1}{2}t}}{t} dt \\
&= \frac{1}{2} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{1}{2}t}}{t} dt - \int_0^\infty \left(\frac{1}{2} - \frac{2}{t} + \frac{e^{\frac{1}{2}t} + 1}{e^t - 1}\right) \frac{e^{-\frac{1}{2}t}}{t} dt \\
&= \frac{1}{2} + \int_0^\infty \left(\frac{1}{t} - \frac{e^{\frac{1}{2}t}}{e^t - 1}\right) \frac{e^{-\frac{1}{2}t}}{t} dt \\
&= \frac{1}{2} + \int_0^\infty \left(\frac{e^{-\frac{1}{2}t}}{t} - \frac{1}{e^t - 1}\right) \frac{dt}{t}
\end{aligned}$$

We therefore obtain, from the second and last equalities,

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{1}{2}t}}{t} dt &= \int_0^\infty \left(\frac{e^{-\frac{1}{2}t}}{t} - \frac{1}{e^t - 1} + \frac{1}{2}e^{-t} - \frac{e^{-t}}{t} + \frac{e^{-t}}{e^t - 1}\right) \frac{dt}{t} \\
&= \int_0^\infty \left(\frac{e^{-\frac{1}{2}t} - e^{-t}}{t} - \frac{1}{2}e^{-t}\right) \frac{dt}{t} \\
&= \int_0^\infty \left(-\frac{d}{dt}\left(\frac{e^{-\frac{1}{2}t} - e^{-t}}{t}\right) - \frac{\frac{1}{2}e^{-\frac{1}{2}t} - e^{-t}}{t} - \frac{e^{-t}}{2t}\right) dt \\
&= \frac{1}{2} + \frac{1}{2}\int_0^\infty \frac{e^{-t} - e^{-\frac{1}{2}t}}{t} dt \\
&= \frac{1}{2} + \frac{1}{2} \log \frac{1}{2}
\end{aligned}$$

Thus we obtain

$$\log\sqrt{\pi} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\log\frac{1}{2} - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-t}}{t}dt$$

$$\int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-t}}{t}dt = 1 - \frac{1}{2}\log(2\pi)$$

and the lemma now follows, i.e.

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log(2\pi) + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-ts}}{t}dt, \Re(s) > 0$$

It is now easy, for example by making use of the Maclaurin expansion for the exponential function, that the last integrand is bounded as the limit

$$\lim_{t\to 0}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{1}{t}$$

exists. So that, if we denote it by M, we have

$$\begin{aligned}\int_0^\infty \left|\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{1}{t}\frac{e^{-ts}}{t}\right| dt &\le M\int_0^\infty |e^{-ts}|dt \\ &= O\left(\frac{1}{|s|}\right)\end{aligned}$$

where $\Re(s) > 1$.

From this the initial proposition follows. □

Proposition 18. *We have, for $T \in \mathbb{R}, T > 0$,*

$$\arg\zeta\left(\frac{1}{2} + iT\right) = O(\log T)$$

as $T \to \infty$.

Lemma 18.1. *Let C be the path consisting of the two broken line segments $2 \to 2 + iT \to \frac{1}{2} + iT, T \in \mathbb{R}, T > 0$. If $\Re\zeta(s)$ has exactly n zeros on C, then*

$$\left|\arg\zeta\left(\frac{1}{2} + iT\right)\right| < \pi n + \frac{\pi}{2}$$

Proof. Since $\zeta(2)$ is real, it follows that $\arg\zeta\left(\frac{1}{2} + iT\right)$ is equal to the change of argument of $\zeta(s)$ as s moves on the path C.

Now we note that if the argument of $\zeta(s)$ is equal to an odd multiple of $\frac{\pi}{2}$, then $\Re\zeta(s) = 0$. At $s = 2$, we have $\arg\zeta(s) = 0$ and the first zero of $\Re\zeta(s)$ on C occurs when $\arg\zeta(s) = \pm\frac{\pi}{2}$. Between two consecutive zeros of $\Re\zeta(s)$, either $\arg\zeta(s)$ varies by 0 or by $\pm\pi$. The lemma immediately follows. □

Lemma 18.2. *$\Re\zeta(s)$ has no zeros on the segment $2 \to 2 + iT, T \in \mathbb{R}, T > 0$.*

Proof. We will in fact prove that $\Re\zeta(2+it), t \in \mathbb{R}$ is never equal to zero. For

$$\begin{aligned}
\Re\zeta(2+it) &= \sum_{n=1}^{\infty} \Re\left(\frac{1}{n^{2+it}}\right) \\
&= \sum_{n=1}^{\infty} \frac{\cos(t\log n)}{n^2} \\
&> 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} \\
&= \frac{3}{4} - \sum_{n=3}^{\infty} \frac{1}{n^2} \\
&> \frac{3}{4} - \int_2^{\infty} \frac{dx}{x^2} \\
&= \frac{1}{4}
\end{aligned}$$

and cannot therefore be 0. □

Lemma 18.3. *For $\Re(s) = \sigma > 0$ and $\Im(s) = t, |t| > 0$, we have*

$$|\zeta(s)| = O(t^2), t \to \infty$$

Proof. This follows easily from the fact that

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s\int_1^{\infty} \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} dx, \Re(s) > 0, s \neq 1$$

which can be easily deduced from the Euler Summation formula. □

Lemma 18.4. *Suppose we have a circle $|s| = r$. Suppose also that we have a nonconstant function $f(s)$ which is holomorphic in the region $|s| \leq r$, while $f(0) = m \neq 0$, and suppose also that $f(s)$ has no zeros on $|z| = r$. Let $M = \max_{|z|=r} |f(s)|$. Let $0 < \rho < r$ and let l denote the number of zeros in the circle $|s| \leq \rho$. Then*

$$\frac{M}{m} > \left(\frac{r}{\rho}\right)^l$$

Proof. In the case where there are no zeros of $f(s)$ in $|s| < r$, we obtain from Cauchy's Integral formula

$$\log f(0) = \frac{1}{2\pi i}\int_{|s|=r} \frac{\log f(s)}{s} ds = \frac{1}{2\pi}\int_0^{2\pi} \log f(Re^{i\theta}) d\theta$$

Suppose instead that there are k zeros, counted with multiplicity, and that they are $a_1, a_2, \ldots, a_k$. Then the function

$$f(s)\prod_{j=1}^{k} \frac{r^2 - \overline{a_j}s}{r(s-a_j)}$$

has no zeros in $|s| \leq r$ and furthermore, on the circle $|s| = r$,

$$\begin{aligned}
\left| f(s) \prod_{j=1}^{k} \frac{r^2 - \overline{a_j} s}{r(s - a_j)} \right| &= |f(s)| \prod_{j=1}^{k} \left| \frac{r^2 - \overline{a_j} s}{r(s - a_j)} \right| \\
&= |f(s)| \prod_{j=1}^{k} \left| \frac{r^2 \overline{s} - \overline{a_j} s \overline{s}}{r^2(s - a_j)} \right| \\
&= |f(s)| \prod_{j=1}^{k} \left| \frac{r^2 \overline{s} - \overline{a_j} r^2}{r^2(s - a_j)} \right| \\
&= |f(s)| \prod_{j=1}^{k} \left| \frac{\overline{s} - \overline{a_j}}{s - a_j} \right| \\
&= |f(s)| \prod_{j=1}^{k} \left| \frac{\overline{s - a_j}}{s - a_j} \right| \\
&= |f(s)|
\end{aligned}$$

giving

$$\log \frac{mr^k}{|a_1||a_2| \dots |a_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta < \log M$$

(the maximum modulus theorem ensures that this is strictly an inequality) from which

$$\frac{M}{m} > \frac{r^k}{|a_1||a_2| \dots |a_k|}$$

The lemma now follows easily. □

Lemma 18.5. *Let n be the number of zeros of $\Re\zeta(s)$ on the line segment $2 + iT \to \frac{1}{2} + iT, T \in \mathbb{R}, T > 0$. Then*

$$n = O(\log T)$$

as $T \to \infty$.

Proof. We define the function $\Phi(s)$ as

$$\Phi(s) = \frac{1}{2}[\zeta(s + Ti) + \zeta(s - Ti)], T \in \mathbb{R}, T > 0$$

Now we may in a similar manner as we have shown previously that $\xi(\overline{s}) = \overline{\xi(s)}$, show that $\zeta(\overline{s}) = \overline{\zeta(s)}$ so that, if $s = \sigma, \sigma \in \mathbb{R}$, we have that

$$\Phi(\sigma) = \Re(\sigma + iT)$$

Let now l denote the number of zeros of the function $\Phi(s)$ in the circle

$$|s - 2| \leq \frac{3}{2}$$

Since $\Phi(s)$ on the real interval $\frac{1}{2} \leq s \leq$ takes on the same values as $\Re\zeta(s)$ on the segment $\frac{1}{2} + Ti \to 2 + Ti$, it now follows that

$$n \leq l$$

We will now show that $l = O(\log T), T \to \infty$ from which the lemma will follow. For this we will use the result of the previous lemma.

We consider the function $\Phi(s+2)$ which translates the point $(2,0)$ onto the origin. We see easily that $\Phi(2) \neq 0$. We take this to be m, in the previous lemma. Next we take $\frac{3}{2} < r \leq 2$. We take $\rho = \frac{3}{2}$. We take T to be sufficiently large large that the points $1 \pm Ti$ lie outside the circle $|s| = 2$ so that $\Phi(s)$ is holomorphic in $|s| \leq 2$. We may assume that there are no zeros of $\Phi(s)$ on the circle $|s-2| = r$ because if there are any, since zeros of holomorphic functions are isolated, we may obtain another circle with r still inside its allowed range by slightly changing the value of r and thereby obtain a circle where no zeros of $\Phi(s)$ lie on the boundary. We let l be the number of zeros inside in the circle $|s| \leq \rho$. Then we have, by the previous lemma,

$$l < \frac{\log \frac{M}{m}}{\log \frac{r}{\rho}}$$

Now, it follows from a previous lemma that

$$|f(s)| = O(T^2), \Re(s) > 0, T \to \infty$$

and therefore,

$$M = O(T^2), T \to \infty$$

whence

$$\log M = O(\log T), T \to \infty$$

and by the inequality just formed, remembering that $m > \frac{1}{4}$ by a previous lemma and is also bounded, we have

$$l = O(\log T), T \to \infty$$

from which the required lemma follows.

Combining this last lemma with the first lemma, the original proposition finally follows. □

Proposition 19. *The number of roots (counted with multiplicities) of $\Xi(t) = 0$ whose real parts lie between 0 and $T, T > 0$ is, up to a **relative** error of the order of magnitude of $\frac{1}{T}$, equal to*

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$$

as $T \to \infty$.

Proof. Let $\mathcal{R}$ be the (positively oriented) rectangle in the complex plane with vertices at the points

$$-1, 2, 2 + iT, -1 + iT$$

We assume that $\xi(s)$ has no zeros on the line $\Im(s) = T$, which is allowed from a consideration that the zeros of holomorphic functions are isolated. Then it follows from what has been shown before that $\xi(s)$ has no zeros on the boundary of $\mathcal{R}$. We then make use of Cauchy's argument principle to deduce that

$$\frac{1}{2\pi i} \oint_{\mathcal{R}} \frac{\xi'(s)}{\xi(s)} ds$$

gives the number of zeros, call this $N(T)$ (counted with multiplicities), of ξ contained in the region bounded by $\mathcal{R}$.

If we briefly substitute $\xi(s) = s$, we have

$$\oint_{\mathcal{R}} \frac{\xi'(s)}{\xi(s)} ds = \oint_{\xi(\mathcal{R})} \frac{dz}{z}$$

so that, by making use of Cauchy's theorem, the integral is seen to equal $2\pi i$ times the winding number (which is an integer here) of the closed curve $\xi(R)$ around the origin.

We may also express this by saying that

$$2\pi N(T) = \frac{1}{i} \oint_{R} \frac{\xi'(s)}{\xi(s)}$$

is equal to the total **change** in the argument of $\xi(s)$ as s travels once around $\mathcal{R}$ counterclockwise (since we are dealing with change of argument we need not take into account the multiple-valued nature of the function $\arg z$).

Before proceeding we note that not only does $\xi(s)$ take on real values on the real axis, as was previously shown, but it also takes on real values on the line $\Im(s) = \frac{1}{2}$; for if in the functional equation

$$\xi(s) = \xi(1 - s)$$

we put

$$s = \frac{1}{2} + \alpha i, \alpha \in \mathbb{R}$$

then we obtain

$$\xi\left(\frac{1}{2} + \alpha i\right) = \xi\left(\frac{1}{2} - \alpha i\right)$$

but we now also have

$$\xi(\overline{s}) = \overline{\xi(s)}$$

so that

$$\xi\left(\frac{1}{2} - \alpha i\right) = \overline{\xi\left(\frac{1}{2} + \alpha i\right)}$$

and hence

$$\xi\left(\frac{1}{2}+\alpha i\right)=\overline{\xi\left(\frac{1}{2}+\alpha i\right)}$$

which is possible if and only if

$$\xi\left(\frac{1}{2}+\alpha i\right), \alpha \in \mathbb{R}$$

is a real number.

Now as s moves from -1 to 2 along the real axis, $\xi(s)$ stays real and so there is no change in its argument. We now also have that the change in argument of $\xi(s)$ as s travels along the path $2 \to 2+iT \to \frac{1}{2}+iT$ is equal to the change in argument of $\xi(s)$ as s travels along the path $\frac{1}{2}+iT \to -1+iT \to -1$; for we have from the properties of ξ already established

$$\xi(\alpha+\beta i)=\overline{\xi(1-\alpha+\beta i)}, \alpha, \beta \in \mathbb{R}$$

so that when s travels along the path $2 \to 2+iT \to \frac{1}{2}+iT$, then ξ will have travelled from the real axis, rotated around the origin a number of times, and come back to the real axis; and from then on, as s travels along the path $\frac{1}{2}+iT \to -1+iT \to -1$, then $\xi(s)$ will continue rotating about the origin in the sense same for the same number of turns (although the shape of the subsequent path will be slightly different) until it comes back to the real axis.

It follows that if we denote by $\mathcal{C}$ the path

$$2 \to 2+iT \to \frac{1}{2}+iT$$

then $\pi N(T)$ will be equal to the change in the argument of $\xi(s)$ as s travels along $\mathcal{C}$. Since

$$\xi(s)=\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)=(s-1)\Gamma\left(\frac{s}{2}+1\right)\pi^{-\frac{s}{2}}\zeta(s)$$

we have that the total change in argument of $\xi(s)$ is equal to the sum of the changes in argument of each of the factors $s-1, \pi^{-\frac{s}{2}}, \Gamma\left(\frac{s}{2}+1\right), \zeta(s)$.

The change in argument of $s-1$ as s moves around $\mathcal{C}$ is equal to $\arg(-\frac{1}{2}+iT)$. But now

$$\begin{aligned} \arg(-\frac{1}{2}+iT)-\frac{\pi}{2} &= \tan^{-1}\frac{1}{2T} \\ &= O\left(\frac{1}{T}\right) \end{aligned}$$

So that the sought change in argument of $s-1$ as s moves around $\mathcal{C}$ is equal to $\frac{\pi}{2}+O\left(\frac{1}{T}\right)$.

Next, the change in argument of $\pi^{-\frac{s}{2}}$ is equal to the change in the imaginary part of $-\frac{s}{2}\log\pi$ as s moves around $\mathcal{C}$, and this is equal to $-\frac{T}{2}\log\pi$.

We now consider the change in argument of $\Gamma\left(\frac{s}{2}+1\right)$ as s moves around $\mathcal{C}$. Again this is the change in the imaginary part of $\log\Gamma\left(\frac{s}{2}+1\right)$ as s moves around $\mathcal{C}$ and this is equal to $\Im\log\Gamma\left(\frac{5}{4}+\frac{iT}{2}\right)$. Now, from the result in a previous proposition,

$$\begin{aligned}\log\Gamma\left(\frac{5}{4}+\frac{iT}{2}\right) &= \left(\frac{3}{4}+\frac{iT}{2}\right)\log\left(\frac{5}{4}+\frac{iT}{2}\right)-\frac{5}{4}-\frac{iT}{2}+\frac{1}{2}\log(2\pi)\\ &\quad+O\left(\frac{1}{\sqrt{(5/4)^2+(T/2)^2}}\right)\\ &= \left(\frac{3}{4}+\frac{iT}{2}\right)\left(\frac{1}{2}\log\left(\frac{25}{16}+\frac{T^2}{4}\right)+i\left(\frac{\pi}{2}-\tan^{-1}\frac{5}{2T}\right)\right)-\frac{5}{4}\\ &\quad-\frac{iT}{2}+\frac{1}{2}\log(2\pi)+O\left(\frac{1}{\sqrt{(5/4)^2+(T/2)^2}}\right)\end{aligned}$$

so that

$$\begin{aligned}\Im\log\Gamma\left(\frac{5}{4}+\frac{iT}{2}\right) &= \frac{3\pi}{8}-\frac{3}{4}\tan^{-1}\frac{5}{2T}-\frac{T}{2}+\frac{T}{4}\log\left(\frac{25}{16}+\frac{T^2}{4}\right)\\ &\quad+O\left(\frac{1}{\sqrt{(5/4)^2+(T/2)^2}}\right)\end{aligned}$$

We therefore have that

$$\Im\log\Gamma\left(\frac{5}{4}+\frac{iT}{2}\right)=\frac{T}{2}\log\frac{T}{2}-\frac{T}{2}+\frac{3\pi}{8}+O\left(\frac{1}{T}\right)$$

as $T\to\infty$.

We finally consider the change in argument of $\zeta(s)$ as s travels along $\mathcal{C}$. This is easily seen to just equal $\arg\zeta(\frac{1}{2}+iT)$(we mean the principal value of the argument). But we have already seen in the previous proposition that $\arg\zeta\left(\frac{1}{2}+iT\right)=O(\log T)$ as $T\to\infty$.

We therefore finally obtain

$$\pi N(T)=\frac{T}{2}\log\frac{T}{2\pi}-\frac{T}{2}+O(\log T)+\frac{7}{8}\pi+O\left(\frac{1}{T}\right)$$

as $T\to\infty$, from which the proposition follows.

We can equivalently say that the number of roots (counted with multiplicities) of

$$\Xi(t)=0$$

whose real parts lie between 0 and T is $\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\log T), T\to\infty$. □

Proposition 20. *We have*

$$\xi(s)=\xi(0)\prod_{\substack{\xi(\rho)=0\\ \Im\rho>0}}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right)$$

Proof. We will do this by showing, in the following lemmas, that there is a (nonzero) constant c such that

$$\xi(s) = c \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{(s-1/2)^2}{(\rho-1/2)^2}\right)$$

For then

$$\xi(0) = c \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{(1/2)^2}{(\rho-1/2)^2}\right)$$

so that

$$\begin{aligned}
\frac{\xi(s)}{\xi(0)} &= \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{(s-1/2)^2}{(\rho-1/2)^2}\right)\left(1 - \frac{(1/2)^2}{(\rho-1/2)^2}\right)^{-1} \\
&= \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \frac{\left(1 - \frac{(s-1/2)}{\rho-1/2}\right)\left(1 + \frac{s-1/2}{\rho-1/2}\right)}{\left(1 + \frac{1/2}{\rho-1/2}\right)\left(1 - \frac{1/2}{\rho-1/2}\right)} \\
&= \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(\frac{\rho-s}{\rho}\right)\left(\frac{\rho+s-1}{\rho-1}\right) \\
&= \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1-\rho}\right)
\end{aligned}$$

from which the proposition follows. □

Lemma 20.1. *The infinite product*

$$\prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1-\rho}\right)$$

is convergent.

Proof. We have

$$\prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1-\rho}\right) = \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{s(1-s)}{\rho(1-\rho)}\right)$$

and the lemma is therefore equivalent to showing that

$$\sum_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \frac{1}{|\rho(1-\rho)|}$$

is convergent. Now

$$\sum_{\substack{\xi(\rho)=0\\ \Im\rho>0}} \frac{1}{|\rho(1-\rho)|} = \sum_{\substack{\xi(\rho)=0\\ \Im\rho>0}} \frac{1}{|\left(\rho-\frac{1}{2}\right)^2-\frac{1}{4}|}$$

and by the limit-comparison test, it suffices to show that

$$\sum_{\substack{\xi(\rho)=0\\ \Im\rho>0}} \frac{1}{\left|\rho-\frac{1}{2}\right|^2}$$

converges. By what has been shown in the previous proposition, we know that as $R \to \infty$, there cannot be more that $R\log R$ zeros in the circle $\left|s-\frac{1}{2}\right| = R$. Now, the zeros of the entire function $\xi(s)$, being isolated, are countably infinite, and suppose we label the zeros $\rho, \Im\rho > 0, \xi(\rho) = 0$ by $\rho_1, \rho_2, \ldots$ in increasing order of moduli (we can consider only zeros above the real axis, due to their symmetry about this axis), then if we let $n = R_n \log R_n$, then there exists a sufficiently large number N such that we have

$$\left|\rho_n - \frac{1}{2}\right| > R_n, n > N$$

and so

$$\begin{aligned}\sum_{n=N}^{\infty} \frac{1}{\left|\rho-\frac{1}{2}\right|^2} &< \sum_{n=N}^{\infty} \frac{1}{R_n^2}\\ &= \sum_{n=N}^{\infty} \left(\frac{\log R_n}{n}\right)^2\end{aligned}$$

Now $\log n = \log R_n + \log\log R_n > \log R_n$ so that

$$\sum_{n=N}^{\infty} \frac{1}{\left|\rho-\frac{1}{2}\right|^2} < \sum_{n=N}^{\infty} \left(\frac{\log n}{n}\right)^2$$

which by a simple comparison test is seen to converge, so that our lemma also follows. □

Lemma 20.2. *We have*

$$|\log\xi(s)| = O(|s\log s|), |s| \to \infty$$

Proof. It was shown in the previous section that

$$\xi(s) = 4\int_1^{\infty} \frac{d\left\{x^{\frac{3}{2}}\psi'(x)\right\}}{dx} x^{-\frac{1}{4}} \cosh\left(\frac{1}{2}\left(s-\frac{1}{2}\right)\log x\right) dx, s \in \mathbb{C}$$

Hence,

$$\xi(s) = 4\sum_{n=0}^{\infty}\left\{\left(s-\frac{1}{2}\right)^{2n}\int_1^{\infty}\frac{d\left\{x^{\frac{3}{2}}\psi'(x)\right\}}{dx}x^{-\frac{1}{4}}\frac{\left(\frac{1}{2}\log x\right)^{2n}}{(2n)!}dx\right\}$$

Now the integrand, and therefore the integral is easily seen to be positive and as in addition, by the maximum modulus theorem the largest value of $|\xi(s)|$ on the disk $\left|s-\frac{1}{2}\right| \le R$ must be $Re^{i\theta}, \theta \in \mathbb{R}$, it follows that the largest value on the circle occurs when $s = \frac{1}{2} + R$. We need therefore only see what happens to $\xi\left(\frac{1}{2}+R\right)$ as $R \to \infty$. Now

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

But we know from a previous proposition that as $|s| \to \infty, \Re(s) > 0$, we have $\Gamma(s) \to s^{s-1/2}\sqrt{2\pi}e^{-s}$. Hence, $\xi\left(\frac{1}{2}+R\right) \to R^R, R \to \infty$, from which the required lemma easily follows. □

Lemma 20.3. *Suppose that s is not equal to any root ρ of $\xi(\rho) = 0$ but otherwise it can take any complex value. Then*

$$\left|\log \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}}\left(1-\frac{(s-1/2)^2}{(\rho-1/2)^2}\right)\right| = O(|s|^2), |s| \to \infty$$

Proof. We easily see that the product converges. Now, eventually, for any fixed s, we have $|s-1/2| < |\rho-1/2|$. We then have

$$\begin{aligned}\left|\log\left(1-\frac{(s-1/2)^2}{(\rho-1/2)^2}\right)\right| &= \left|\sum_{n=1}^{\infty}\frac{(s-1/2)^{2n}}{n(\rho-1/2)^{2n}}\right| \\ &\le \sum_{n=1}^{\infty}\left|\frac{s-1/2}{\rho-1/2}\right|^{2n} \\ &= \left|\frac{s-1/2}{\rho-1/2}\right|^2\left(1-\left|\frac{s-1/2}{\rho-1/2}\right|^2\right)^{-1} \\ &= \frac{|s-1/2|^2}{|\rho-1/2|^2-|s-1/2|^2}\end{aligned}$$

For sufficiently large real N,

$$\left|\sum_{\substack{\xi(\rho)=0 \\ \Im\rho>N}}\log\left(1-\frac{(s-1/2)^2}{(\rho-1/2)^2}\right)\right|$$

is therefore convergent by a previous lemma and is $O(|s|^2), |s| \to \infty$. The convergence implies that the infinite sum of logarithms must differ from the logarithm of the infinite product by a finite multiple of $2\pi i$ (?) and so the lemma follows. $\square$

Lemma 20.4. *We have*

$$\xi(s) = c \prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{(s-1/2)^2}{(\rho-1/2)^2}\right)$$

for a nonzero constant c.

Proof. It easily follows from the previous lemmas that

$$\left|\log \frac{\xi(s)}{\prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{(s-1/2)^2}{(\rho-1/2)^2}\right)}\right| = O\left(\left|\frac{\log s}{s}\right|\right), |s| \to \infty$$

so that

$$\Re\log \frac{\xi(s)}{\prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{(s-1/2)^2}{(\rho-1/2)^2}\right)} = O\left(\left|\frac{\log s}{s}\right|\right), |s| \to \infty$$

and from this we are able to conclude that for arbitrarily small $\epsilon > 0$, we can always find sufficiently large $R > 0$ such that $\Re f(s) < \epsilon|s|^2, |s| > R$. Now, we see that the infinite product in the denominator has the same zeros as the numerator; moreover by a similar method used in a previous proposition, we see that the product is entire, so that, since the numerator has already been shown to be entire, it follows that the fraction is an entire function of s with no zeros. Besides, when s is replaced by $1-s$, or equivalently $s-1/2$ by $1/2-s$, the function defined by this fraction remains unchanged, due to the functional equation for ξ; in other words, the fraction is an entire function of $s-1/2$. Let us denote

$$f(s) = \Re\log \frac{\xi(s)}{\prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(1 - \frac{(s-1/2)^2}{(\rho-1/2)^2}\right)}$$

Let us in addition define another function

$$F\left(s - \frac{1}{2}\right) = f(s) - f\left(\frac{1}{2}\right)$$

Since $F(0) = 0$, it follows by the Borel-Carathéodory theorem that, in any disk $|s| \le R, R > 0$, we have

$$\left|F\left(s - \frac{1}{2}\right)\right| \le \frac{2Mr}{(R-r)}, |s| < r$$

for $0 < r < R$, where $M = \max_{|s|=R} F\left(s - \frac{1}{2}\right) = \max_{|s|\leq R} F\left(s - \frac{1}{2}\right)$; we see that M can be made smaller that ϵR^2 for every arbitrarily small $\epsilon > 0$ for sufficiently large R.

Now since F has no singularities, $F(0) = 0$, and since F is an even function of $s - \frac{1}{2}$, it follows that we can represent it by a Taylor series with an infinitely large radius of convergence

$$F\left(s - \frac{1}{2}\right) = \sum_{n=1}^{\infty} a_{2n} \left(s - \frac{1}{2}\right)^{2n}$$

We will now show that $a_{2n} = 0, n \in \mathbb{N}$. We put $z = s - \frac{1}{2}$ and get $F(z) = \sum_{n=1}^{\infty} a_{2n} z^n$ where, for example, from Cauchy' s Integral formulas,

$$a_{2n} = \frac{1}{2\pi i} \oint_C \frac{F(z)}{z^{2n+1}} dz, C : |z| = R/2, R > 0$$

so that

$$|a_{2n}| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{2^{2n} |F\left(\frac{1}{2} Re^{i\theta}\right)|}{R^{2n}} d\theta$$

But by what has been said before, if we select $r = R/2$, then

$$\frac{2^{2n} \left|F\left(\frac{1}{2} Re^{i\theta}\right)\right|}{R^{2n}} \leq \frac{2^{2n+1}\epsilon}{R^{2n-2}}$$

for arbitrarily small ϵ for sufficiently large R, so that a_{2n} must be zero. The lemma easily follows now. $\square$

1.3 The explicit formula

Proposition 21. *Let*

$$\pi(x) = \sum_{\substack{p \leq x \\ p \ prime}} 1$$

Define

$$F(x) = \frac{\pi(x+0) + \pi(x-0)}{2}$$

Finally define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} F(x^{\frac{1}{n}})$$

Then we have

$$f(x) = \frac{1}{2\pi i} \int_{\sigma - \infty i}^{\sigma + \infty i} \frac{\log \zeta(s)}{s} x^s ds, \sigma \in \mathbb{R}, \sigma > 1$$

where, when x is a discontinuity we should take the left side to be $\frac{f(x+0)+f(x-0)}{2}$. Also the limits of the complex integral mean that we are integrating along the line with real part σ in the complex $s - plane$.

Proof.

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \Re(s) > 1$$

$$\begin{aligned}
\log \zeta(s) &= \log \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \\
&= - \sum_{p \text{ prime}} \log \left(1 - \frac{1}{p^s}\right) \\
&= \sum_{n=1}^{\infty} \sum_{p \text{ prime}} \frac{1}{n} p^{-ns} \\
\frac{1}{s} \log \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{p \text{ prime}} \int_{p^n}^{\infty} x^{-s-1} dx
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{n} \int_1^{\infty} F(x^{\frac{1}{n}}) x^{-s-1} dx &= \frac{1}{n} \int_1^{\infty} F(x) x^{-ns-n} n x^{n-1} dx \\
&= \int_1^{\infty} F(x) x^{-ns-1} dx \\
&= \int_1^{2} F(x) x^{-ns-1} dx + \int_2^{3} F(x) x^{-ns-1} dx + \int_3^{5} F(x) x^{-ns-1} dx + \dots \\
&= \int_2^{3} x^{-ns-1} dx + 2 \int_3^{5} x^{-ns-1} dx + 3 \int_5^{7} x^{-ns-1} dx + \dots \\
&= \sum_{p \text{ prime}}^{\infty} \int_p^{\infty} x^{-ns-1} dx \\
&= \frac{1}{n} \sum_{p \text{ prime}}^{\infty} \int_{p^n}^{\infty} x^{-s-1} dx \\
\therefore \frac{1}{s} \log \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} F(x^{\frac{1}{n}}) x^{-s-1} dx
\end{aligned}$$

Now for any real number $b > 1$, we have

$$\sum_{n=1}^{\infty} \int_1^{b} \frac{1}{n} F(x^{\frac{1}{n}}) x^{-s-1} dx = \int_1^{b} \sum_{n=1}^{\infty} \frac{1}{n} F(x^{\frac{1}{n}}) x^{-s-1} dx$$

because the sum is actually then a finite sum. Since the left side also converges as $b \to \infty$, the right side must also converge to the same sum. This gives

$$\frac{1}{s} \log \zeta(s) = \int_1^{\infty} f(x) x^{-s-1} dx, \Re s > 1$$

We now make the substitution $x \to e^u$ to obtain

$$\frac{1}{s}\log\zeta(s) = \int_0^\infty f(e^u)e^{-us}du$$

or, since $f(x) = 0, 0 \le x \le 1$,

$$\frac{1}{s}\log\zeta(s) = \int_{-\infty}^\infty f(e^u)e^{-us}du, \Re s > 1$$

We can also rewrite this as

$$\frac{1}{s}\log\zeta(s) = \frac{1}{2\pi}\int_{-\infty}^\infty \left(2\pi f(e^u)e^{-\sigma u}\right)e^{-iut}du, \sigma = \Re(s) > 1, t = \Im s$$

Now the non-periodic function $2\pi f(e^u)e^{-\sigma u}$ of u satisfies the Dirichlet conditions of Fourier analysis and so we may apply the Fourier inversion theorem to obtain

$$2\pi e^{-\sigma x}\frac{f(e^{x+0}) + f(e^x - 0)}{2} = \int_{-\infty}^\infty \frac{1}{s}\log\zeta(s)e^{itx}dt$$

So that

$$\begin{aligned}\frac{f(e^{x+0}) + f(e^x - 0)}{2} &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{1}{s}\log\zeta(s)e^{\sigma+itx}dt \\ \frac{f(y+0) + f(y-0)}{2} &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{1}{s}\log\zeta(s)y^s dt \\ &= \frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{1}{s}\log\zeta(s)y^s ds, \sigma > 1\end{aligned}$$

The limits of the complex integral have the meaning already indicated above. For simplicity we will write

$$f(x) = \frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{1}{s}\log\zeta(s)x^s ds, \sigma > 1$$

where it is understood that when x is a discontinuity, the left side means $\frac{f(x+0)+f(x-0)}{2}$. □

Proposition 22. *(Main Result) We have*

$$f(x) = \mathrm{li}(x) - \sum_{\substack{\xi(\rho)=0 \\ \Im\rho>0}} \left(\mathrm{li}(x^\rho) + \mathrm{li}(x^{1-\rho})\right) + \int_x^\infty \frac{dt}{t(t^2-1)\log t}dt + \log\xi(0), x > 0$$

where $f(x)$ is defined as in the previous proposition and where

$$\mathrm{li}(x) = \lim_{\epsilon\to 0+}\left(\int_0^{1-\epsilon}\frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t}\right), x > 1$$

Proof. We have

$$\begin{aligned}
\xi(s) &= \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \\
&= (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s) \\
\zeta(s) &= \xi(s)(s-1)^{-1}\pi^{\frac{s}{2}}\frac{1}{\Gamma\left(\frac{s}{2}+1\right)} \\
&= \xi(0)\prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right)(s-1)^{-1}\pi^{\frac{s}{2}}\frac{1}{\Gamma\left(\frac{s}{2}+1\right)}
\end{aligned}$$

$$\log\zeta(s) = \log\xi(0) + \log\prod_{\substack{\xi(\rho)=0 \\ \Im\rho>0}}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right) - \log(s-1) + \frac{s}{2}\log\pi - \log\Gamma\left(\frac{s}{2}+1\right)$$

If we replace this into the result of the previous proposition and attempt to integrate termwise, we see that some of the integrals to which we are led do not converge. (This is not a strange phenomenon: for example we have previously found that $\int_0^\infty \left(\frac{1}{e^x-1}-\frac{e^x}{x}\right)dx = \gamma$, but if we attempt termwise integration in this, we will be led to divergent integrals). To get around this difficulty, we will integrate by parts the integral

$$\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{1}{s}\log\zeta(s)x^s ds, \sigma > 1$$

We recall that for $s = \sigma + it, \sigma > 1$, we have as previously

$$\log\zeta(s) = \log\zeta(\sigma) + i\arg\zeta(\sigma+it) = O(\log t)$$

so that

$$\lim_{t\to\pm\infty}\frac{\log\zeta(s)}{s}x^s = 0$$

so that integration by parts will yield

$$f(x) = -\frac{1}{2\pi i\log x}\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{d\left\{\frac{\log\zeta(s)}{s}\right\}}{ds}x^s ds, \sigma > 1, x > 1$$

so that

$$f(x) = \frac{1}{2\pi i \log x}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{1}{s^2}\log \xi(0) x^s ds$$

$$- \frac{1}{2\pi i \log x}\int_{\sigma+\infty i}^{\sigma+\infty i} \frac{d\left\{\frac{1}{s}\prod_{\substack{\xi(\rho)=0\\ \Im\rho>0}}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right)\right\}}{ds} x^s ds$$

$$+\frac{1}{2\pi i \log x}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{d\left\{\frac{1}{s}\log(s-1)\right\}}{ds} x^s ds - \frac{1}{2\pi i \log x}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{d\left\{\frac{1}{s}\log\Gamma\left(\frac{s}{2}+1\right)\right\}}{ds} x^s ds,$$

$$\sigma > 1$$

The proposition will then follow from the following lemmas. □

Lemma 22.1.

$$\frac{1}{2\pi i \log x}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{1}{s^2}\log \xi(0) x^s ds = \log \xi(0), \sigma > 1$$

Proof. For this term, given that the integral already converged before we integrated by parts previously, we can reverse the integration by parts to obtain

$$\frac{\log \xi(0)}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{x^s}{s} ds, \sigma > 1$$

To complete the result, we will evaluate a more general case which will be useful in the following lemmas:

$$\begin{aligned}
\frac{1}{\sigma+it-\beta} &= \int_0^\infty e^{-x(\sigma+it-\beta)}dx, \sigma > \Re(\beta)\\
&= \int_0^\infty e^{-ixt}e^{x(\beta-\sigma)}dx\\
&= \int_0^\infty e^{x(\beta-\sigma)}\cos(xt) - ie^{x(\beta-\sigma)}\sin(xt)dx
\end{aligned}$$

Assuming that we can use Fourier's integral theorem, we have

$$\begin{aligned}
e^{x(\beta-\sigma)} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(xt)}{\sigma+it-\beta}dx, x > 0\\
\text{and } -ie^{x(\beta-\sigma)} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(xt)}{\sigma+it-\beta}dt, x > 0\\
\text{giving } 2\pi e^{x(\beta-\sigma)} &= \int_{-\infty}^{\infty}\frac{e^{itx}}{a+it-\beta}dt, \sigma > 0, x > 0
\end{aligned}$$

When we substitute $x = \log y$ and we replace $s = \sigma + it$, we can rewrite this as

$$\frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{1}{s-\beta}x^s ds = x^\beta, \sigma > 0, x > 1$$

The present lemma then follows by putting $\beta = 1$. □

Lemma 22.2.

$$-\frac{1}{2\pi i\log x}\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{d\left\{\frac{1}{s}\log\Gamma\left(\frac{s}{2}+1\right)\right\}}{ds}x^s ds=\int_x^\infty\frac{1}{x^2-1}\frac{dx}{x\log x},\sigma>1$$

Proof. We firstly recall that from a result which we have previously proved

$$\begin{aligned}\Gamma\left(\frac{s}{2}+1\right)&=\lim_{m\to\infty}\frac{m!m^{\frac{s}{2}}}{\left(\frac{s}{2}+1\right)\left(\frac{s}{2}+2\right)\dots\left(\frac{s}{2}+m\right)}\\&=\lim_{m\to\infty}\frac{m^{\frac{s}{2}}}{\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)\dots\left(\frac{s}{2m}+1\right)}\\\log\Gamma\left(\frac{s}{2}+1\right)&=\log\lim_{m\to\infty}\frac{m^{\frac{s}{2}}}{\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)\dots\left(\frac{s}{2m}+1\right)}\end{aligned}$$

Assuming that the result obtained in interchanging the limit and the logarithm, and then converting the logarithm of a product into a sum of logarithms is valid, we have

$$\begin{aligned}-\log\Gamma\left(\frac{s}{2}+1\right)&=\lim_{m\to\infty}\left\{\sum_{n=1}^{m}\log\left(1+\frac{s}{2n}\right)-\frac{s}{2}\log m\right\}\\-\frac{1}{s}\log\Gamma\left(\frac{s}{2}+1\right)&=\lim_{m\to\infty}\left\{\sum_{n=1}^{m}\frac{1}{s}\log\left(1+\frac{s}{2n}\right)-\frac{1}{2}\log m\right\}\\-\frac{d\left\{\frac{1}{s}\log\left(\frac{s}{2}+1\right)\right\}}{s}&=\frac{d}{ds}\lim_{m\to\infty}\left\{\sum_{n=1}^{m}\frac{1}{s}\log\left(1+\frac{s}{2n}\right)-\frac{1}{2}\log m\right\}\end{aligned}$$

When we now assume that the differentiation and summation signs may be interchanged, we have, assuming convergence of the integrals,

$$-\frac{d\left\{\frac{1}{s}\log\left(\frac{s}{2}+1\right)\right\}}{s}=\sum_{n=1}^{\infty}\frac{d\left\{\frac{1}{s}\log\left(1+\frac{s}{2n}\right)\right\}}{ds}$$

$$\frac{1}{2\pi i\log x}\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{d}{ds}\left\{\frac{1}{s}\log\Gamma\left(\frac{s}{2}+1\right)x^s ds\right\}=-\frac{1}{2\pi i\log x}\int_{\sigma-\infty i}^{\sigma+\infty i}\sum_{n=1}^{\infty}\frac{d\left\{\frac{1}{s}\log\left(1+\frac{s}{2n}\right)\right\}}{ds}x^s ds$$

Assuming that we may interchange the summation and integration signs, we have

$$\frac{1}{2\pi i\log x}\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{d}{ds}\left\{\frac{1}{s}\log\Gamma\left(\frac{s}{2}+1\right)x^s ds\right\}=-\frac{1}{2\pi i\log x}\sum_{n=1}^{\infty}\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{d\left\{\frac{1}{s}\log\left(1+\frac{s}{2n}\right)\right\}}{ds}x^s ds$$

(To be continued) □

Lemma 22.3.

$$\frac{1}{2\pi i\log x}\int_{\sigma-\infty i}^{\sigma+\infty i}\frac{d\left\{\frac{1}{s}\log\left(s-1\right)\right\}}{ds}x^s ds=\mathrm{li}(x),\sigma>1$$

Proof. To be supplied. □

Lemma 22.4.

$$\frac{1}{2\pi i \log x}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{d\left\{\frac{1}{s}\log\prod_{\substack{\xi(\rho)=0\\ \Im\rho>0}}\left(1+\frac{s(1-s)}{\rho(1-\rho)}\right)\right\}}{ds} x^s ds = \sum_{\substack{\xi(\rho)=0\\ \Im\rho>0}} \left\{\mathrm{li}(x^\rho)+\mathrm{li}(x^{1-\rho})\right\}, \sigma>1$$

Proof. To be supplied. □

2 Some useful Books

- *Complex Analysis* by Ahlfors
- *A Course of Modern Analysis* by Whittaker and Watson
- *Riemann's Zeta Function* by H.M. Edwards