SMALL GAPS WITHIN PRIMES

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Abstract. Primes are important because they are fundamental units of all integers, and there has been many questions about primes, such as the explicit formula for the number of primes less than a certain number. In this paper, I will introduce basic definitions and theorems necessary to understand the gaps between primes. Afterwards, I will prove ... and ..., which are significant theorems regarding the gaps between primes.

1. Introduction

As said before, prime numbers are building blocks of all numbers, which is why they are frequently studied in number theory. Before I go into talking about gaps within primes, I will introduce one of the most important theorems regarding prime numbers, which is known as the prime number theorem.

Theorem 1.1. *Let $\zeta(x)$ be the exact number of primes less than x.* $\lim_{x\to\infty} \frac{\zeta(x)}{x/log(x)} = 1$.

In simple terms, this means that as x gets infinitely large, the number of primes less than x is $x/\log(x)$. Also, a log in this paper means a natural log. As a result of the prime number theorem, following important corollary regarding the gaps between primes can be established.

Corollary 1.2. *The average gap between a consecutive pair of primes is approximately* $\log(x)$ *as x becomes infinitely large.*

There are $\frac{x}{\log(x)}$ number of primes less than x according to the prime number theorem. If we assume that the biggest prime less than x is not too small compared to x, then the sum of all gaps between primes should be close to x. Because there are $\frac{x}{\log(x)}$ number of primes, we can evaluate $\lim_{x\to\infty} \frac{x}{x/\log(x)-1} = \log(x)$ to show that the average gap between the primes is approximately $\log(x)$. We also have following conjectures that are important regarding the gaps within primes as they investigate the smallest and largest gaps within primes.

2. preliminaries

Conjecture 2.1. *There are infinitely many twin primes.*

Definition 2.2. Twin primes are a prime that is either 2 less than the next biggest problem or 2 greater than the previous prime.

As an example, $11, 13, 41, 43$ are all twin primes.

Conjecture 2.3. *Let p^n denote the nth prime.* $\sup_{p_n \le x}(p_{n+1} - p_n) = (\log(x))^{2+o(1)}$.

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The conjecture above is called Cramér's conjecture, and it is a weak form of Cramér's conjecture. To clarify some important notations that will be used in this paper, below are some definitions to keep in mind.

Definition 2.4. For some $A \subset \mathbb{R}$, $\inf A = \{m : m \leq a\}, \forall a \in A$.

In other words, the infimum of a set is the greatest lower bound of that set.

Definition 2.5. For some $A \subset \mathbb{R}$, $\sup A = \{m : m \geq a\}, \forall a \in A$.

Similarly, the supremum of a set is the smallest upper bound of the set.

Definition 2.6. Let $f : \mathbb{R} \to \mathbb{R}$. $g(x) = O(f(x))$ if there exists some constant c, d such that $g(x) \leq c * f(x)$ for all $x > d$.

To give an example, if we let $g(x) = x^2 + 2x + 1$, then $g(x) = O(x^2)$ because $2x^2$ is greater than $g(x)$ after a certain point.

Definition 2.7. Let $f : \mathbb{R} \to \mathbb{R}$. $g(x) = o(f(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.

$o(x)$ acts as an extreme upper bound. To give an example, $x = o(x^2)$.

Conjecture 2.8. *Let $L_1, L_2, \ldots, L_k$ be integral and linear functions with $L_i(n) = a_i n + b_i$ such that for every prime p there is an integer n_p with $\prod_{i=1}^{k} L_i(n_p)$ coprime to p. Then, there are infinitely many integers n such that all of $L_1(n), L_2(n), \ldots, L_k(n)$ are all primes.*

This conjecture is called the prime k-tuple conjecture. I will talk about GPY Sieve method, a technique essential to prove a weaker version of the prime k-tuple conjecture, in the next section. Then, I will prove the weaker version of the prime k-tuple conjecture.

3. GPY Sieve Method

Sieve methods were originally developed to find all primes. GPY Sieve Method was created by Goldson, Pintz, and Yildrium to study small gaps within primes. Here is an outline of the GPY Sieve method.

(1) Choose a probability measure ω on integers in $[x, 2x]$.
(2) Calculate the expected number of he functions $L_i(n)$ which are prime at n, if n is randomly chosen with probability $\omega(n)$.
(3) If the expected value is at least m, there is some $n \in [x, 2x]$ such that at least m of the integrable functions are prime.
(4) If the 3rd step holds true for a large x, then there are infinitely many possible values of n.

4. A Weaker Version of the Prime k-tuple Conjecture

Below is the weaker version of the prime k-tuple conjecture we're trying to prove:

Theorem 4.1. *Let $L_1, L_2, \ldots, L_k$ be integrable linear functions $L_i(n) = a_i n + b_i$, such that for every prime p, there is an integer n_p with $\prod_{i=1}^{k} L_i(n_p)$ coprime to p. Then, there is a constant $c > 0$ such that there are infinitely many integers n with at least $c \log(k)$ of the k integrable functions being primes.*

With modified GPY Sieve Method, we can choose a function $G(t)$ satisfying conditions that make the method convenient, which are

$$k\int_0^\infty t^2 G(t)^2 dt \to 0$$

and

$$k(\int_0^\infty G(t))^2 dt \to \infty$$

as $k \to \infty$. To satisfy those conditions, we let

$$G(t) = \begin{cases} \frac{\sqrt{k\log(k)}}{1+tk\log(k)} & t < k^{-3/4} \\ 0 & \text{otherwise} \end{cases}$$

Using this function in the Sieve method, we can find that

$$E\#\{i : L_i(n)\text{prime}\} \geq (c\log(k) + o(1))\frac{\log(R)}{\log(x)}$$

and this shows how GPY Sieve Method can be used to prove theorem 4.1.

References

[1] James Maynard. Gaps between primes. pages 1–6, 2019.

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