Consequences of the Riemann Hypothesis

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1 Introduction

The Riemann hypothesis has multiple implications, which include theorems that allow us to understand prime distribution. Some of the examples include Koch's result that gave precise bounds to the error of the prime number theorem (1901), and Schoenfeld's bound for the error of Chebyshev's second function (1976). In this paper we will be focusing on another result on prime distribution in an interval, (Theorem 1.3) as proven by Dudek [1](2014)

Conjecture 1 (Riemann Hypothesis). *If* $0 < \Re(s) < 1$ *and* $\zeta(s) = 0$*, then* $\Re(s) = \frac{1}{2}$

Where $\pi(x)$ is the number of primes $\leq x$, and $\mathrm{li}(x)$ is the logarithmic integral function, assuming the Riemann hypothesis, we can get the following bound:

Theorem 1.1 (Koch). *Assuming Riemann hypothesis true,* $\forall\ x \geq 22657$ *the error can be bounded by:*

$$|\pi(x) - li(x)| < \frac{1}{8\pi}\sqrt{x}\log(x)$$

Theorem 1.2 (Schoenfeld). *Assuming Riemann hypothesis true,* $\forall\ x \geq 273.2$*, the following holds:*

$$|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x}\log^2(x)$$

Where $\psi(x)$ is Chebyshev's second function, the Riemann hypothesis can allow us to deduce a bound for the error, getting the above result by Schoenfeld

Theorem 1.3 (Dudek). *Assuming Riemann hypothesis true,* $\forall\ x \geq 2$*, there is a prime* p *satisfying the below:*

$$x - \frac{4}{\pi}\sqrt{x}\log(x) < p \leq x$$

In order to prove this, we will first prove a few results on the way. Let us look at the Riemann von-Mangoldt explicit formula, for the function $\psi(x) = \sum_{n\leq x} \Lambda(n)$, where Λ is the von-Mangoldt function.

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - x^{-2})$$

Where $x > 0$ and $x \notin \mathbb{Z}$ and the sum is over all non-trivial zeros $\rho = \beta + i\gamma$.

Let us integrate both sides of the equation on the interval $(2, x)$ to get

$$\int_2^x \psi(t)\,dt = \frac{x^2 - 2^2}{2} + \sum_\rho \left(\frac{x^{\rho+1}}{\rho(\rho+1)} - \frac{2^{\rho+1}}{\rho(\rho+1)}\right) - x\log 2\pi - \frac{1}{2}\int_2^x \log(1-t^{-2})\,dt \tag{1}$$

Defining a new weighted sum to be $\psi_1(x)$,

$$\psi_1(x) = \sum_{n\le x}(x-n)\Lambda(n) = \int_2^x \psi(t)\,dt$$

Combining this with equation 1 we get that

$$\psi_1(x) = \frac{x^2}{2} + \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\log 2\pi + \epsilon(x)$$

where

$$|\epsilon(x)| < 2 + \left|\sum_\rho \frac{2^{\rho+1}}{\rho(\rho+1)}\right| + \frac{1}{2}\left|\int_2^x \log(1-t^{-2})\,dt\right|$$

We can evaluate the integral to $\log(\frac{16}{27})$ and we can get an estimate on the sum assuming the Riemann Hypothesis, giving us:

$$\left|\sum_\rho \frac{2^{\rho+1}}{\rho(\rho+1)}\right| < 2^{3/2}\left|\sum_\rho \frac{1}{|\rho|^2}\right|$$

This sum has an explicitly known value, as shown in Davenport's work [3] , giving us the following lemma

Lemma 1.4.

$$\psi_1(x) = \frac{x^2}{2} + \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\log 2\pi + \epsilon(x) \tag{2}$$

where

$$|\epsilon(x)| < \frac{12}{5}$$

Let us consider prime number distribution in smaller intervals by defining the weighted function over the interval $(x-h, x+h)$

$$\omega(n) = \begin{cases} 1 - |n-x|/h & : x-h < n < x+h \\ 0 & : \text{otherwise} \end{cases} \tag{3}$$

Notice how this can be imagined as a weighted ratio of the distance of n around an interval over x of length h.

After some simplifications and manipulations, we get the following identity

$$\sum_{n} \Lambda(n)\omega(n) = \frac{1}{h}\Big(\psi_1(x+h) - 2\psi_1(x) + \psi_1(x-h)\Big) \tag{4}$$

Putting in the relations from Lemma 1.4 to equation 4, we get the following lemma

Lemma 1.5. *Let $x > 0$ and $h > 0$*

$$\sum_{n} \Lambda(n)\omega(n) = h - \frac{1}{h}\Sigma + \epsilon(h) \tag{5}$$

where

$$\Sigma = \sum_{\rho} \frac{(x+h)^{\rho+1} - 2x^{\rho+1} + (x-h)^{\rho+1}}{\rho(\rho+1)}$$

and

$$|\epsilon(h)| < \frac{48}{5h}$$

We will split the sum Σ into two parts, based on the magnitude of the imaginary part of γ of $\rho = \beta + \gamma i$

$$\Sigma = \Sigma_1 + \Sigma_2$$

where Σ_1 runs over all zeros where $\gamma < \alpha x/h$ for some $\alpha > 0$, and Σ_2 runs over the rest of the zeros.

We can rewrite the Σ_1 as an integral in the following way

$$\int_{x-h}^{x+h} (h - |x-u|)u^{\rho-1}du$$

This integral can now be bounded

$$\left| \int_{x-h}^{x+h} (h - |x-u|)u^{\rho-1}du \right| < \frac{1}{\sqrt{x-h}} \int_{x-h}^{x+h} (h - |x-u|)du = \text{`} \frac{h^2}{\sqrt{x-h}}$$

Hence we get that

$$\Sigma_1 \leq \frac{h^2}{\sqrt{x-h}} \sum_{|\gamma| < \alpha x/h} 1$$

$$= \frac{2h^2}{\sqrt{x-h}} N(\alpha x/h)$$

where $N(\alpha x/h)$ is the number of zeros ρ for which $0 < \beta < 1$ and $0 < \gamma < T$. We add a factor of 2, since we're taking the absolute value of γ.

According to Trudgian's paper [2], we have the bound:
$\forall\, T > 15$,

$$N(T) < \frac{T \log T}{2\pi} \tag{6}$$

Hence, putting this in the existing equation for Σ_1, we get, for $\alpha x/h > 15$,

$$\Sigma_1 < \frac{2h^2}{2\pi\sqrt{x-h}}(\alpha x/h)\log(\alpha x/h) = \frac{\alpha x h}{\pi\sqrt{x-h}}\log(\alpha x/h) \tag{7}$$

Similarly, we will try to bound Σ_2 the following way, by the using the Riemannn hypothesis

$$\begin{aligned} |\Sigma_2| &< 4(x+h)^{3/2} \sum_{|\gamma|>\alpha x/h} \frac{1}{\gamma^2} \\ &= 8(x+h)^{3/2} \sum_{\gamma>\alpha x/h} \frac{1}{\gamma^2} \end{aligned}$$

Lemma 1 (ii) in Skewes' work [4] tells that

$$\sum_{\gamma \geq T} \frac{1}{\gamma^2} < \frac{1}{2\pi}\frac{\log T}{T}$$

Therefore, we have

$$\Sigma_2 < \frac{4h(x+h)^{3/2}}{\pi\alpha x}\log(\alpha x/h)$$

Combining both the bounds in Lemma 1.5, we get:

$$\sum_n \Lambda(n)\omega(n) > h - \frac{1}{h}(|\Sigma_1| + |\Sigma_2|) - \frac{48}{5h} \tag{8}$$

$$= h - \left(\frac{\alpha x}{\pi\sqrt{x-h}} + \frac{4(x+h)^{3/2}}{\pi\alpha x}\right)\log(\alpha x/h) - \frac{48}{5h} \tag{9}$$

Since h is $o(x)$, we get that the factor of the log term is asymptotic to

$$\left(\frac{\alpha}{\pi} + \frac{4}{\pi\alpha}\right)\sqrt{x}$$

which will be minimum at $\alpha = 2$, simplifying our inequality to give

$$\sum_n \Lambda(n)\omega(n) > h - \frac{2}{\pi}\left(\frac{x}{\sqrt{x-h}} + \frac{(x+h)^{3/2}}{x}\right)\log(2x/h) - \frac{48}{5h}$$

And we have

$$\psi(x+h) - \psi(x-h) = \sum_{x-h<n\le x+h} \Lambda(n)$$

Now, since $(1 - |n-x|/h) < 1$ for all n in this sum, our smart construction of $\omega(n)$ will now allow us to bound it the following way

$$\psi(x+h) - \psi(x-h) > \sum_{n} \Lambda(n)\omega(n) \tag{10}$$

$$> h - \frac{2}{\pi}\left(\frac{x}{\sqrt{x-h}} + \frac{(x+h)^{3/2}}{x}\right)\log(2x/h) - \frac{48}{5h} \tag{11}$$

Let us consider Chebyshev's θ-function defined as

$$\theta(x) = \sum_{p\le x} \log p$$

According to Schoenfeld (Theorem 14 and eq 5.5) [5], we can get the following bounds on the functions:
$\forall\ x > 121$,

$$0.98\sqrt{x} < \psi(x) - \theta(x) < 1.11\sqrt{x} + 3x^{1/3}$$

Putting this into our existing inequality for $\psi(x+h) - \psi(x-h)$, we get

$$\begin{aligned}\sum_{x-h<p\le x+h} \log p &= \theta(x+h) - \theta(x-h)\\ &> h - \frac{2}{\pi}\left(\frac{x}{\sqrt{x-h}} + \frac{(x+h)^{3/2}}{x}\right)\log(2x/h) - 11.1\sqrt{x+h}\\ &\quad - 3(x+h)^{1/3} + 0.98\sqrt{x-h} - \frac{48}{5h}\end{aligned}$$

If we keep $h = d\sqrt{x}\log x$, the leading term in the equation will be asymptotic to

$$\left(d - \frac{2}{\pi}\right)\sqrt{x}\log x + \frac{4}{\pi}\sqrt{x}\log\log x$$

Now we observe that for $d > \frac{2}{\pi}$, the right hand side of the equation will be positive, hence the value of the sum $\sum_{x-h<p\le x+h}\log p$ will also be positive, showing the existance of at least one prime in the interval

$$(x - d\sqrt{x}\log x, x + d\sqrt{x}\log x]$$

Thus, we choose the minimum value of d, i.e $d = \frac{2}{\pi}$. We will substitute x with $x + d\sqrt{x}\log x$, and by the argument of monotonicity, we have, for all $x \ge 65000$, there is at least one prime in the interval

$$(x - \frac{4}{\pi}\sqrt{x}\log x, x]$$

$\forall\ x \geq 65000 + \frac{2}{\pi}\sqrt{65000}\log(65000) \approx 66799$

We can verify the statement for the rest of the values of $x > 2$ using computational tools such as Mathematica.

Hence, we have proven Theorem 1.3. $\square$

Let us look at how this interval actually looks like in numbers. Here is how the interval can be visualised at $x = 10,000$.

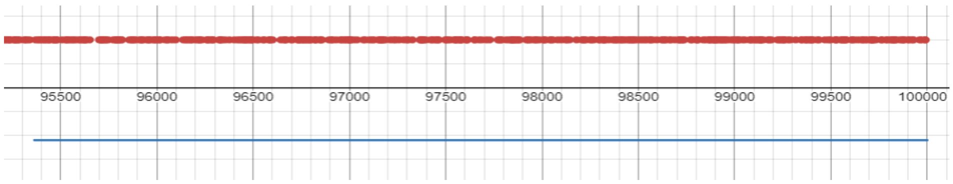

Figure 1: Prime number distribution: Primes until $10,000$ (in red) and the interval $(x - \frac{4}{\pi}\sqrt{x}\log x, x]$ at $x = 10,000$ (in blue). (Graphed in Desmos)

The theorem will prove to be more useful as the numbers increase to larger amounts.

Conclusion

This is just one theorem of prime distribution that we looked at, there are many more such theorems that are a consequence of the Riemann Hypothesis, and help us better understand prime distribution. Understanding prime distribution is critical and has applications even outside of mathematics, in the world of hashing and cryptography. Some other consequences of the RH and GRH include the theorems regarding large and small gaps between primes and Primality tests that run in polynomial time.

References

[1] A. Dudek *On the Riemann Hypothesis and the difference between primes*, The Australian National University (2014).

[2] T. S. Trudgian. *An improved upper bound for the argument of the Riemann zeta-function on the critical line.* Mathematics of Computation (2012).

[3] H. Davenport. *Multiplicative Number Theory.* Springer, Berlin (1980).

[4] S. Skewes. *On the difference* $\pi(x) - li(x)$ (II). Proceedings of the London Mathematical Society (1955).

[5] L. Schoenfeld. *Sharper bounds for the Chebyshev functions* $\theta(x)$ *and* $\psi(x)$. II. Mathematics of Computation (1976).