Chebyshev's Bias

Trevor Johnson

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Chebyshev's Bias is the name for a phenomenon that has been observed, where for most x, the number of primes less than or equal to x congruent to 3 (mod 4) is greater than the number of primes less than or equal to x congruent to 1 (mod 4). Chebyshev's Bias was first observed in 1853 by the Russian mathematician Pafnuty Chebyshev. It should be noted that the difference between the two quantities is $o(x)$. However, it is thought that the Bias continues as x goes to ∞, although it has not been completely proven yet. As of the time of this paper being written, all proofs of the Chebyshev's Bias require a stronger form of the Riemann hypothesis. The proof in this paper assumes the Grand Riemann Hypothesis.

Definition 1. *The prime-counting function $\pi(x; n, a)$ is the number of primes less than or equal to x congruent to a (mod n).*

Definition 2. *$P_{q:a_1,\ldots,a_r}(x)$ is the set of $x \geq 2$ such that $\pi(x; q, a_1) > \cdots > \pi(x; q, a_r)$.*

Definition 3. *Let $\bar{\delta}(P) = \limsup_{x\to\infty} \frac{1}{\ln X} \int_{t\in P\cap[2,x]} \frac{dt}{t}$ and $\underline{\delta}(P) = \liminf_{x\to\infty} \frac{1}{\ln X} \int_{t\in P\cap[2,x]} \frac{dt}{t}$ where P is a set. The logarithmic density of the set P is $\delta(P) = \bar{\delta}(P) = \underline{\delta}(P)$ whenever $\bar{\delta}(P) = \underline{\delta}(P)$. For example, $\delta(\mathbb{R}) = 1$.*

Definition 4. *$E_{q:a_1,\ldots,a_r}(x)$ is a vector-valued function defined as $\frac{\ln x}{\sqrt{x}} \times (\phi(q)\pi(x, q, a_1) - \pi(x), \ldots, \phi(q)\pi(x, q, a_r) - \pi(x))$*

Definition 5. *Let χ be a Dirichlet character mod q. $\psi(x, \chi) = \sum_{n\leq x} \chi(n)\Lambda(n)$.*

Theorem 1. *If χ is not the principal Dirichlet character, $x \geq 2$, and $X \geq 1$, then*

$$\psi(x,\chi) = -\sum_{|\gamma_X| \leq X} \frac{x^\rho}{\rho} + O\left(\frac{x\ln^2(xX)}{X} + \ln x\right)$$

where $\rho = \beta_\chi + i\gamma_\chi$ goes over the zeroes $L(s,\chi)$ in $0 < \Re(s) < 1s$. The proof for this was omitted for space. It can be found in Multiplicative Number Theory by H. Davenport from page 115 to 120. [1]

Remark 1. *By the Riemann Hypothesis, $\beta_\chi = \frac{1}{2}$, so*

$$\psi(x,\chi) = -\sqrt{x}\sum_{|\gamma_X| \leq X} \frac{x^{i\gamma_\chi}}{\frac{1}{2} + i\gamma_\chi} + O\left(\frac{x\ln^2(xX)}{X} + \ln x\right)$$

Definition 6.

$$c(q,a) = -1 + \sum_{\substack{b^2 \equiv a \ (mod\ q) \\ 0 \leq b \leq q-1}} 1$$

Definition 7.

$$\begin{aligned} \psi(x,q,a) &= \sum_{\substack{n \leq x \\ n \equiv a \ (mod\ q)}} \Lambda(n) = \frac{1}{\phi(q)} \sum_{\chi\ mod\ q} \bar{\chi}(a) \sum_{n \leq x} \Lambda(n)\chi(n) \\ &= \frac{1}{\phi(q)} \sum_{\chi\ mod\ q} \bar{\chi}(a), \psi(x,\chi) \end{aligned}$$

Theorem 2 (Dirichlet's Theorem for Progressions)**.** *Let a and m be co-prime integers. There are an infinite number of primes p such that $p \equiv a$ $(mod\ m)$.*

Proof Sketch. Only a sketch of the proof is provided in this paper. The full proof can be found in Chapter 6 of *A Course in Arithmetic* [3].

Let P_a be the set of primes p such that $p \equiv a \pmod{m}$.
Let $f_\chi(s) = \sum_{p \nmid m} \frac{\chi(p)}{p^s}$ where χ is a Dirichlet character mod m.
Show that f_χ diverges as $s \to 1$ iff χ is the principal Dirichlet character.
Show that $\sum_\chi \chi(a^{-1}p) = \phi(m)$ if $a^{-1}p \equiv 1 \pmod{m}$ and 0 otherwise.

Analyse the function $g_a(s) = \sum_\chi \chi(a)^{-1} f_\chi(s)$ to show that the number of primes congruent to a mod m is infinite. □

Theorem 3 (Kronecker-Weyl Equidistribution Theorem)**.** *The real numbers* 1, v_1, $v_2, \ldots,$ *and* v_d *are rationally independent iff the line* $t(v_1, \ldots, v_d)$ *where* $t \in \mathbb{R}$ *is equidistributed on the d-dimensional torus.*

Proof. The proof for this theorem can be found in *A note on the Kronecker–Weyl equidistribution theorem*[4]. $\square$

Theorem 4. $E_{q:a_1,\ldots,a_r}$ *has a limiting distribution* $\mu_{q:a_1,\ldots,a_r}$*. In other words, for any continuous bounded function* f *in* $\mathbb{R}^r$, $\lim_{X\to\infty} \frac{1}{\ln X} \int_2^X f(E_{q:a_1,\ldots,a_r}(x)) \frac{dx}{x} = \int_{\mathbb{R}^r} f(x)\, d\mu_{q:a_1,\ldots,a_r}(x)$.

Lemma 1.

$$E_{q:a}(x) = -c(q,a) + \sum_{\chi \neq \chi_0} \bar{\chi}(a) \frac{\phi(x,\chi)}{\sqrt{x}} + O\left(\frac{1}{\ln x}\right)$$

where χ_0 *is the principal Dirichlet character.*

Proof. Let $\theta(x,q,a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \ln p.$

$$\pi(x,q,a) = \int_2^x \frac{d\theta(t,q,a)}{\ln t}$$

By Dirichlet's theorem for progressions,

$$\psi(x,q,a) = \theta(x,q,a) + \left(\sum_{b^2 \equiv a \pmod q} 1\right) \frac{\sqrt{x}}{\phi(q)} + O\left(\frac{\sqrt{x}}{\ln x}\right)$$

which means that

$$\begin{aligned}\int_2^x \frac{d\theta(t,q,a)}{\ln t} =& \frac{1}{\phi(q)}\int_2^x \frac{d\psi(t)}{\ln t} + \frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}\bar{\chi}(a)\int_2^x \frac{d\psi(t,\chi)}{\ln t} - \frac{1}{\phi(q)}\left(\sum_{b^2\equiv a \pmod q} 1\right)\frac{\sqrt{x}}{\ln x}\\ &+ O\left(\frac{\sqrt{x}}{\ln^2 x}\right)\\ =& \frac{1}{\phi(q)}\left(\pi(x)+\frac{\sqrt{x}}{\ln x}\right) + \frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}\bar{\chi}(a)\frac{\psi(x,\chi)}{\ln x} - \frac{1}{\phi(q)}\left(\sum_{b^2\equiv a \pmod q} 1\right)\frac{\sqrt{x}}{\ln x}\\ &+ O\left(\sum_{\chi\neq\chi_0}\left|\int_2^x \frac{\psi(t,\chi)}{t\ln^2 t}\right| + \frac{\sqrt{x}}{\ln^2 x}\right)\\ =& \frac{\pi(x)}{\phi(q)} + \frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}\bar{\chi}(a)\frac{\psi(x,\chi)}{\ln x} - \frac{c(q,a)}{\phi(q)}\frac{\sqrt{x}}{\ln x} + O\left(\sum_{\chi\neq\chi_0}\left|\int_2^x \frac{\psi(t,\chi)}{t\ln^2 t}\right| + \frac{\sqrt{x}}{\ln^2 x}\right)\end{aligned}$$

Let $G(x,\chi) = \int_2^x \psi(t,\chi)\,dt$. Integrating shows that

$$G(x,\chi) = -\sum_{\gamma_\chi} \frac{x^{3/2+i\gamma_\chi}}{(1/2+i\gamma_\chi)(3/2+i\gamma_\chi)} + O(x\ln x)$$

The asymptotic formula for the number of zeroes (Davenport p. 101 [1]) shows that:

$$\#\{|\gamma_\chi| \le T\} = \frac{T}{\pi}\log\frac{qT}{2\pi} - \frac{T}{\pi} + O(\ln T + \ln q)$$

which means that $G(x,\chi) \ll_q x^{3/2}$. Integrating the large equation gives

$$\pi(x,q,a) - \frac{\pi(x)}{\phi(q)} = -\frac{c(q,a)}{\phi(q)}\frac{\sqrt{x}}{\ln x} + \frac{1}{\phi(q)\ln x}\sum_{\chi\neq\chi_0}\bar{\chi}(a)\psi(x,\chi) + O\left(\frac{\sqrt{x}}{\ln^2 x}\right)$$

This is the end of the proof. □

Remark 2. *Combining the equations from Remark 1 and Lemma 1 gives that, for $T \ge 1$ and $2 \le x \le X$,*

$$E_{q:a}(x) = -c(q,a) - \sum_{\chi\neq\chi_0}\bar{\chi}(a)\sum_{|\gamma_\chi|\le T}\frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi} + \epsilon_a(x,T,X)$$

where

$$\epsilon_a(x,T,X) = -\sum_{\chi\neq\chi_0} \bar{\chi}(a) \sum_{T\le|\gamma_\chi|\le X} \frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi} + O_q\left(\frac{\sqrt{x}\ln^2 X}{X} + \frac{1}{\ln x}\right)$$

Lemma 2.

$$\int_{\ln 2}^{Y} |\epsilon_a(e^y,T,e^Y)|^2\,dy \ll_q Y\frac{\ln^2 T}{T} + \frac{\ln^3 T}{T}$$

Proof.

$$\begin{aligned}
\int_{\ln 2}^{Y} |\epsilon_a(e^y,T,e^Y)|^2\,dy &\ll \int_{\ln 2}^{Y} \left| \sum_{\chi\neq\chi_0} \bar{\chi}(a) \sum_{T\le|\gamma_\chi|\le e^Y} \frac{e^{iy\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi} \right| dy + O(1) \\
&= \sum_{\substack{\chi\neq\chi_0\\ \lambda\neq\chi_0}} \sum_{\substack{T\le|\gamma_\chi|\le e^Y\\ T\le|\gamma_\lambda|\le e^Y}} \bar{\chi}(a)\lambda(a) \int_{\ln 2}^{Y} \frac{e^{iy(\gamma_\chi-\gamma_\lambda)}}{\left(\frac{1}{2}+i\gamma_\chi\right)\left(\frac{1}{2}-i\gamma_\lambda\right)}\,dy + O(1) \\
&\ll_q \sum_{\substack{\chi\neq\chi_0\\ \lambda\neq\chi_0}} \sum_{\substack{T\le|\gamma_\chi|\le\infty\\ T\le|\gamma_\lambda|\le\infty}} \frac{1}{|\gamma_\chi||\gamma_\lambda|} \min\left\{Y, \frac{1}{|\gamma_\chi-\gamma_\lambda|}\right\}
\end{aligned}$$

Using the asymptotic formula for the number of zeroes and comparing the sum to the equation

$$\int_T^\infty \int_T^\infty \frac{\ln x \ln y}{xy} \min\left\{Y, \frac{1}{|y-x|}\right\} dx\,dy$$

gives an error bound of $O\left(Y\frac{\ln^2 T}{T} + \frac{\ln^3 T}{T}\right)$. □

Lemma 3. *For each T there is a probability measure v_T over $\mathbb{R}^r$ such that*

$$v_T(f) = \int_{\mathbb{R}^r} f(x)\,dv_T(x) = \lim_{Y\to\infty} \frac{1}{Y} \int_{\ln 2}^{Y} f(E^{(T)}(y))\,dy$$

for all continuous bounded function f where

$$E_j^{(T)}(y) = -c(q,a_j) - \sum_{\chi\neq\chi_0} \bar{\chi}(a_j) \sum_{|\gamma_\chi|\le T} \frac{e^{iy\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi}$$

and

$$E^{(T)}(y) = (E_1^{(T)}(y),\ldots,E_r^{(T)}(y))$$

Proof. Let $\gamma_1, \ldots, \gamma_N$ be the numbers such that $0 \le \gamma_j \le T$ and $\frac{1}{2} + i\gamma_j$ is a zero of $L(s, \chi)$. $E^{(T)}(y)$ can be written as

$$E^{(T)}(y) = 2\Re\left(\sum_{l=1}^{N} b_l e^{iy\gamma_l}\right) + b_0$$

where

$$b_0 = -(c(q, a_1), \ldots, c(q, a_r))$$

and

$$b_l = -\left(\frac{\bar{\chi}_l(a_1)}{\frac{1}{2} + i\gamma_l}, \ldots, \frac{\bar{\chi}_l(a_r)}{\frac{1}{2} + i\gamma_l}\right)$$

Define the function $g(y_1, \ldots, y_N)$ over $T^N = \mathbb{R}^N/\mathbb{Z}^N$ to be

$$g(y_1, \ldots, y_N) = f\left(2\Re\left(\sum_{l=1}^{N} b_l e^{2\pi i\gamma_l}\right) + b_0\right)$$

Notice that g is continuous and

$$f(E^{(T)}(y)) = g\left(\frac{\gamma_1 y}{2\pi}, \ldots, \frac{\gamma_r y}{2\pi}\right)$$

Let A be the topological closure in T^N of the subgroup

$$\Gamma(y) = \left\{\frac{\gamma_1 y}{2\pi}, \ldots, \frac{\gamma_N y}{2\pi}\,\middle|\, y \in R\right\}$$

A is a torus. $\Gamma(y)$ is equidistributed in A by the Kronecker-Weyl Theorem. $g|_A$ is continuous on A, so

$$\lim_{Y\to\infty} \frac{1}{Y} \int_{\ln 2}^{Y} f(E^{(T)}(y))\, dy = \int_A g(a)\, da$$

where da is the normalized Haar measure on A. Therefore,

$$v_T(f) = \int_A g(a)\, da$$

satisfies the first half of the theorem. From the definition of $E^{(T)}(y)$,

$$|E_j^{(T)}(y)| \ll \sum_{|\gamma_\chi| \le T} \frac{1}{|\gamma_\chi| + 1} \ll_q \ln^2 T$$

□

Proof of Theorem 4. Let f be a function such that there exists a constant c_f where $|f(x) - f(y)| \leq c_f|x - y|$ for all x and y.

$$\begin{aligned}\frac{1}{Y}\int_{\ln 2}^{Y} f(E(y))\,dy &= \frac{1}{Y}\int_{\ln 2}^{Y} f(E^{(T)}(y) + \varepsilon^{(T)}(y))\,dy \\ &= \frac{1}{Y}\int_{\ln 2}^{Y} f(E^{(T)}(y))\,dy + O\left(\frac{c_f}{Y}\int_{\ln 2}^{Y} |\varepsilon^{(T)}(y)|\,dy\right)\end{aligned}$$

where $\varepsilon^{(T)}(y) = E_{q:a_1,\ldots,a_r}(y) - E^{(T)}(y)$. By Lemma 2,

$$\begin{aligned}&\frac{1}{Y}\int_{\ln 2}^{Y} f(E^{(T)}(y))\,dy + O\left(\frac{c_f}{Y}\int_{\ln 2}^{Y} |\varepsilon^{(T)}(y)|\,dy\right) \\ &\qquad = \frac{1}{Y}\int_{\ln 2}^{Y} f(E^{(T)}(y))\,dy + O\left(\frac{c_f}{\sqrt{Y}}\left(\int_{\ln 2}^{Y} |\varepsilon^{(T)}(y)|\,dy\right)^{\frac{1}{2}}\right) \\ &\qquad = \frac{1}{Y}\int_{\ln 2}^{Y} f(E^{(T)}(y))\,dy + O\left(c_f\left(\frac{\ln T}{\sqrt{T}} + \frac{\ln^2 T}{Y\sqrt{T}}\right)\right)\end{aligned}$$

Using Lemma 3 and letting $Y \to \infty$ gives

$$\begin{aligned}v_T(f) - O\left(\frac{c_f \ln T}{\sqrt{T}}\right) &\leq \liminf \frac{1}{Y}\int_{\ln 2}^{Y} f(E_{q:a_1,\ldots,a_r}(y))\,dy \\ &\leq \limsup \frac{1}{Y}\int_{\ln 2}^{Y} f(E_{q:a_1,\ldots,a_r}(y))\,dy \\ &\leq v_T(f) + O\left(\frac{c_f \ln T}{\sqrt{T}}\right)\end{aligned}$$

Letting $T \to \infty$ shows that both limits must converge to the same number, so $\mu(f)$ exists. □

Theorem 5. *The Fourier transform of* $\mu_{q:a_1,\ldots,a_r}$ *is*

$$\hat{\mu}_{q:a_1,\ldots,a_r}(\xi_1,\ldots,\xi_r) = e^{i\sum_{j=1}^{r} c(q,a_j)\xi_j} \prod_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}} \prod_{\gamma_\chi>0} J_0\left(\frac{2|\sum_{j=1}^{r}\chi(a_j)\xi_j|}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right)$$

where J_0 *is the Bessel function*

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{z}{2})^2 m}{(m!)^2}$$

The proof for this has also been omitted for space. It can be found in section 3 of the article on Chebyshev Bias in Experimental Mathematics Vol. 3.[2]

Theorem 6. *Prime numbers are biased towards being congruent to* 3 *(mod* 4*) rather than* 1 *(mod* 4*).*

Proof. The first factor in Theorem 5 causes the mean of $\mu_{q:a_1,\ldots,a_r}$ to be $-(c(q,a_1),\ldots,c(q,a_r))$. This results in the Chebyshev bias. □

References

[1] Davenport, H. and Montgomery, H.L. (2013) *Multiplicative Number Theory*, Springer New York.

[2] Rubinstein, Michael and Sarnak, Peter. (1994) *Chebyshev Bias*, Taylor & Francis.

[3] Serre, J. P. (1973). *A Course in Arithmetic*. In Graduate texts in mathematics.

[4] Beck, J., Chen, W. W. L., Yang, Y. (2024, March 29). *A note on the Kronecker–Weyl equidistribution theorem.* https://arxiv.org/html/2404.00077v1